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# On the impact of cross-ownership in a common property renewable resource oligopoly*

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**Abstract**

This paper examines a dynamic game of exploitation of a productive asset by agents who subsequently sell the outcomes of their endeavours in an oligopolistic market where a subset of the oligopolists owns a share in each other's profits. A Markov Perfect Nash Equilibrium of the game is constructed and used to analyze the impact of cross-ownership on the equilibrium production strategies, the steady state resource stocks, the profitability of cross-ownership, and social welfare. We show that there exists an interval of resource stocks for which a symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are unprofitable in the corresponding static equilibrium framework. Moreover, we demonstrate that cross-ownership may not only lead to a higher market output and social welfare in the short run, but also a higher steady-state stock, industry production, and social welfare in the long run. Thus, antitrust authorities should be cautious in ruling in the renewable resource industries.

**Keywords:** Cross-ownership, Renewable Resource, Closed-loop, Cournot Competition, Oligopoly, Shareholdings, Resource Stock, Antitrust

**JEL Codes:** L13, L41, Q2, C73, D43

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# 1 Introduction

In a static Cournot oligopoly homogenous-product model where competing firms engage in rival cross-shareholdings and participate in various forms of cooperation,[1] the static theory predicts that cross-ownership participants will reduce their output while non-participants expand their production, leading to a lower industry output and consumer surplus, but higher producer surplus and ultimately a lower welfare (Reynolds and Snapp, 1986; Bresnahan and Salop, 1986; Farrell and Shapiro, 1990; O'Brien and Salop, 2000; Dai, Benchekroun and Long, 2022). Moreover, there exists a cross-ownership paradox analogous to the seminal merger paradox first proposed by Salant, Switzer and Reynolds (1983) that the seemingly profitable rival cross-shareholdings may turn out to be unprofitable (Dai, Benchekroun and Long, 2022). However, these static results may not necessarily carry over to the case of a common property renewable resource industry where the resource stock, if left unexploited, reproduces itself naturally at a rate that depends on the size of the stock (Benchekroun, 2008; Benchekroun and Gaudet, 2015).

In this paper, we examine the impact of cross-ownership on the equilibrium production strategies, profitability of cross-shareholdings, and social welfare in the context of a natural resource oligopoly. To do that, we use a dynamic game model where firms exploit the common property productive asset and compete as Cournot rivals in the output market (Benchekroun, 2008; Colombo and Labrecciosa, 2018), but a subset of firms engage in rival cross-shareholdings. Instead of making the usual assumption that firms seek to maximize the discounted sum of their own profits, we take into account the complex economic ties that exist in an industry characterized by rival cross-shareholdings, where the aggregate profits of a firm include not only the stream of profits generated from its own operations but also a share in its competitors' aggregate profits due to its direct and indirect ownership stakes in these firms (Flath, 1992; Gilo, Moshe and Spiegel, 2006). Such cross-ownership activities are particularly prevalent in the renewable resource industries. For example, A Matis report commissioned by Seafish and the Grimsby seafood cluster sheds light on the intricate web of connections and dependencies in ownership within the largest seafood companies in Iceland.[2] In addition, the implementation of a tradeable quota system has incentivized large cross-ownership in the New Zealand fishing industry as firms try to circumvent restrictions on the maximum quotas individuals can own.[3] Abundant evidence has also been documented for the cooperation between fishermen who typically live in small communities and behave cooperatively by exchanging information, sharing costs and dividing labour, which entitles them to a share of the profits from other fishermen's catches(Colombo and Labrecciosa, 2018).

[1]See Benchekroun, Dai and Long (2022) and Huse, Ribeiro and Verboven (2024) for examples.
[2]See https://www.seafish.org/document/?id=4d67242e-b529-43b9-81aa-477de5a9133e.
[3]See https://www.fao.org/3/Y2498E/y2498e0e.htm.

We perform our analysis in the context of a differential game (see Dockner et al. (2000), Long (2010) and Başar and Zaccour (2018) for concepts and applications), and we focus on closed-loop or feedback strategies, where firms' strategies are production rules that depend both on time and the asset's stock.[4] A Markov Perfect Nash Equilibrium of the game is then characterized and used to contrast with the case without cross-ownership.

We first show that there exist scenarios such that the cross-ownership participants may increase their output and the non-participants may lower their production as a result of cross-ownership in the short run. The former can occur for some range of large initial resource stocks when all the firms in the industry engage in cross-shareholdings, while the number of firms is larger than two and the ownership stake is not large enough. The latter can occur for some range of small initial resource stocks when only a subset of firms participate in cross-ownership. These results are quite counterintuitive and are in sharp contrast with the static oligopoly theory and cross-ownership theory. Indeed, when rival firms participate in cross-ownership, they have an incentive to compete less aggressively as one firms gain may come at the loss of the other firms in which it has shareholdings. As such, each cross-ownership participant will reduce their output, but in terms of strategic substitutes in Cournot competition, firms that do not participate in cross-ownership will always respond by expanding their production. However, in our context where an oligopoly exploits a common property productive asset, there is an additional channel through which cross-ownership influences firms' extraction rates, beyond the typical static "market power" mechanism due to reduced competition in the output market. That is, cross-ownership also affects how firms interact with each other at the resource level.

When the initial resource stock is abundant enough, firms will behave as if they are not constrained by the resource stock, and thus resource scarcity plays no role. However, when the resource stock falls below a certain threshold, a positive resource rent arises, affecting firms' production strategies. The impact of rival cross-shareholdings on the resource rent manifests as a negative effect for relatively large stocks and a positive effect for relatively small stocks. In the former case, cross-owners will only slightly reduce their production due to a relatively low shareholding in the output market, but this results in a sufficiently abundant stock of the asset, which in turn reduces the marginal valuation of the resource stock by each cross-ownership participant. A smaller resource rent thus incentivizes each cross-owner to expand its production, which outweighs the production reduction brought by cross-ownership at the output market. In the latter case, the asset remains sufficiently scarce for the relatively low levels of stocks, leading to an increase in the marginal valuation of the resource stock for each cross-ownership non-participant. A higher resource rent thus provides an incentive for the outsiders to reduce their production, which dominates the static

[4]For a more detailed discussion on the comparison between open-loop and closed-loop strategies in renewable resource games, see e.g., Colombo and Labrecciosa (2018).

effect of production expansion induced by cross-ownership.

We then establish that these scenarios may indeed materialize, since firms may find it profitable to engage in rival cross-shareholdings. We thus conduct a detailed profitability analysis to understand the private incentives driving competing firms to participate in cross-ownership arrangements. Specifically, we examine the profitability of cross-ownership in the context of a common property renewable resource oligopoly and compare it to the static case. We show that the cross-ownership paradox does not necessarily carry over to the case of a renewable resource industry. There always exists an interval of resource stocks for which a symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are strictly unprofitable in the corresponding static equilibrium framework. The main intuition behind this result lies in the common property nature of renewable asset exploitation. When multiple players share a common resource, the rate at which it is exploited (the decision variable) is intrinsically linked to the available stock of the resource (the state variable). This interdependence means that any action by one player that alters the stock level will have a direct impact on the decisions made by all other players in the industry. But no such link exists in the corresponding static oligopoly with cross-ownership, where any given rate of production can be sustained forever. In the static game, the outsiders always respond aggressively by expanding their production when insiders decrease their output due to their ownership stakes. However, in the dynamic setting, the outsiders might respond more cautiously or even reduce their output in some instances. This moderated response occurs because, within certain stock ranges, cross-ownership can lead to an increased valuation by each player for the marginal unit of remaining resource stock. A consequence is that there is always an interval of initial stocks such that the profitability of cross-ownership is always positive.

Moreover, we demonstrate that there exists a specific range of resource stocks in which not only is cross-ownership between rival firms profitable, but it also increases industry production. One direct implication of this result is that consumer surplus will increase following the profitable cross-shareholdings, which, at the same time, boosts industry profits, leading to a higher overall welfare in the short run. This outcome sharply contradicts the traditional static theory, according to which cross-ownership always leads to a welfare loss in the absence of any efficiency gains (Reynolds and Snapp, 1986). This aspect holds significant relevance for discussions surrounding competition policies, as there is a growing call for more stringent regulations of these non-controlling minority shareholdings that are currently subject to a very lenient approach by antitrust authorities. Our findings thus suggest that competition authorities should be cautious when ruling in the renewable resource sector, as cross-ownership may turn out to be welfare-improving.

Finally, we delve into the effects of cross-ownership on steady-state resource stocks, industry outputs, profitability, and social welfare in the long run. Our analysis shows that cross-ownership results in a larger steady-state level of the productive asset's stock, regardless of the initial resource stock. Additionally, we demonstrate that cross-shareholdings between rival firms can result in an increase in the industry's output at the stationary equilibrium when the implicit growth rate falls below a certain threshold or the initial resource stock is small enough. This result presents a stark contrast to the static theory, which traditionally asserts that cross-ownership leads to a decrease in industry output. The key insight here is that in our dynamic framework with a common productive asset, cross-ownership influences the industry's exploitation rate through two main channels: the output market and the interaction at the resource level. The former represents the traditional mechanism by which reduced competition in the output market due to ownership links leads to a decrease in industry output. The latter, unique to the renewable resource industry, suggests that cross-ownership can lead to a larger steady-state stock of the asset, thereby enabling greater industry extraction. This interaction at the resource level thus significantly alters the dynamics of industry output and ultimately leads to increased industry production in the long run, challenging the conventional static perspective.

Furthermore, we demonstrate that the above-mentioned scenarios could occur, as firms will find it profitable to engage in cross-shareholdings in the transition to the steady state of the stocks. Consequently, the long-run expansion of industry production becomes a viable prospect, suggesting the potential for an increase in consumer surplus in the long run as a result of cross-ownership. We then show that producer surplus is also higher at the stationary equilibrium for these scenarios, which implies that welfare can increase due to cross-shareholdings in the long run as well. Therefore, antitrust authorities should exercise caution when regulating renewable resource industries, as strict policies that restrict cooperation among users of common-pool renewable resources could ultimately harm consumers and society. Unintentionally, these measures might produce the exact opposite effect of what is intended.

Our paper contributes to several strands of literature. The first one is on the growing literature on cross-ownership (Reynolds and Snapp, 1986; Bresnahan and Salop, 1986; Farrell and Shapiro, 1990; Flath, 1991, 1992; Malueg, 1992; O'Brien and Salop, 2000; Dietzenbacher, Smid and Volkerink, 2000; Gilo, Moshe and Spiegel, 2006; Brito, Cabral and Vasconcelos, 2014; Brito, Ribeiro and Vasconcelos, 2014; Brito et al., 2018; Brito, Ribeiro and Vasconcelos, 2018; Dai, Benchekroun and Long, 2022; Huse, Ribeiro and Verboven, 2024). However, most of these studies have focused mainly on the potential anticompetitive effects induced by cross-ownership, i.e., unilateral effects and coordinated effects, and failed to explain why and to what extent firms want to engage in cross-shareholdings with the only exception of Dai, Benchekroun and Long (2022). We add to the literature by providing the first analysis that investigates the impact of cross-ownership in a common property renewable resource industry, and we also demonstrate that the results obtained in the traditional static theory do not necessarily carry over to the renewable resource industry. While Dai, Benchekroun and Long (2022) consider a nonrenewable resource oligopoly with each firm owning a private resource stock, we consider a common property renewable resource in this paper. Moreover, as opposed to Dai, Benchekroun and Long (2022) where each firm's strategy consists of an extraction path, in this paper each firm chooses an extraction policy that is stock dependent, thus ensuring subgame perfectness of the equilibrium we characterize.

Our paper also contributes to the large game-theoretic literature on the exploitation of renewable resources. Previous studies have focused either on the case in which agents behave non-cooperatively (Levhari and Mirman, 1980; Reinganum and Stokey, 1985; Karp, 1992; Dockner and Sorger, 1996; Dawid and Kopel, 1997; Sorger, 1998; Benchekroun, 2003, 2008; Sandal and Steinshamn, 2004; Colombo and Labrecciosa, 2013a,b, 2015; Benchekroun and Gaudet, 2015; Benchekroun and Van Long, 2016) or the case where agents act cooperatively (Benhabib and Radner, 1992; Kopel and Szidarovszky, 2006; Colombo and Labrecciosa, 2018). Despite such a well-established literature in resource economics, no previous studies have studied how ownership links between any rival firms may affect the use of a renewable resource. By distinguishing the situation in which all the firms in the industry participate and only a subset of firms engage in rival cross-shareholdings, we illustrate how cross-ownership might affect the exploitation of a common property productive asset and its impact on social welfare.

Our paper is closely related to Colombo and Labrecciosa (2018), but differs significantly in several aspects. While they focus primarily on cooperative strategies in a duopoly context, we examine a broader oligopoly setting, exploring scenarios where all firms engage in cross-shareholding and cases where only a subset of firms do so. This more general approach has allowed us to analyze the effects of cross-ownership in a more varied and realistic market structure, providing some new insights into its impact on market behaviour and competition. In addition, we delve into the private incentives of rival firms to participate in cross-ownership and demonstrate that there exist situations where the industry output, consumer surplus, producer surplus and social welfare may increase following the profitable rival cross-shareholdings in both the short run and long run.

Finally, our research also adds to the ongoing policy discussions on how to regulate cross-ownership. While horizontal mergers generally face considerable antitrust scrutiny and often encounter opposition from antitrust authorities, non-controlling minority shareholdings tend to escape similar levels of examination. As highlighted by Gilo (2000) and Gilo, Moshe and Spiegel (2006), these types of arrangements often enjoy a de facto exemption from antitrust liability or receive minimal attention from antitrust agencies. According to Nain and Wang (2018), fewer than 1% of minority acquisitions are challenged by the Federal Trade Commission (FTC) or the Department of Justice (DOJ), and even fewer are blocked outright. In the European Union and many other jurisdictions, antitrust authorities do not even have the competence to investigate such cases.[5] Therefore, firms might consider cross-ownership a more appealing corporate strategy and opt for it disproportionately, knowing that it generally lacks legal accountability (Jovanovic and Wey, 2014; Dai, Benchekroun and Long, 2022). This trend is particularly concerning given the minimal antitrust enforcement against non-controlling minority shareholdings, allowing firms to reap the benefits of cooperation with competitors while avoiding the scrutiny typically associated with horizontal mergers. Regarding this, there have been growing calls for more stringent regulations to limit rival-cross-shareholdings. Our findings, however, suggest that antitrust authorities should be careful in ruling in renewable resource industries, as cross-ownership may actually lead to a higher social welfare. Applying a "per se illegal" antitrust policy is misleading, as strict application of such a policy to renewable resources neglects the dynamics of the common resource stock that should be explicitly taken into account (Adler, 2004; Deacon, 2012; Benchekroun and Gaudet, 2015; Colombo and Labrecciosa, 2018).

The remainder of the paper is structured as follows. Section 2 presents the model. Section 3 investigates the profitability of cross-ownership and looks at the impact of cross-ownership in the short run. Section 4 investigates the effects of cross-ownership in the long run. Finally, Section 5 concludes.

## 2 The model and preliminary analysis

Let $S$ denote the stock of some renewable assets, for instance, a fish population. We assume that in the absence of exploitation, the stock of the asset evolves according to the following dynamics (see e.g., Benchekroun (2008) and Benchekroun and Gaudet (2015)):

$$\frac{dS}{dt} = F(S) = \begin{cases} \delta S & \text{for } S \leq S_y \\ \delta S_y \left( \frac{\bar{S}-S}{\bar{S}-S_y} \right) & \text{for } S > S_y \end{cases}, \quad S(0) = S_0 \geq 0,$$

where $S_0$ is the initial stock of the asset, $\delta > 0$ is the intrinsic growth rate, $\bar{S}$ is the maximum carrying capacity and $\delta S_y$ is the maximum sustainable yield of the asset. This specification of $F(S)$ can be thought of as a linearization of the classical logistic growth function in natural resource economics (Clark, 2010; Conrad, 2010). When the stock is very small (i.e., $S \leq S_y$), there is no habitat constraint and the asset grows at an exponential rate; however, beyond $S_y$, the asset grows at a decreasing rate facing limited availability of food and space. Without loss of generality, we set $\bar{S} = 1$ in what follows.

The access to this asset is shared by $J = \{1, 2, \cdots, n\}$ firms, indexed by $j$, where each firm exploits the asset to produce an output to sell in an oligopolistic market. For simplicity, we assume that one unit of the asset is transformed into one unit of the

[5] See Fotis and Zevgolis (2016) for more discussions.

output at zero cost. Let $q_j(t)$ denote firm $j$'s output at time $t$, and the inverse demand function for the output at time $t$ is given by

$$p(t) = a - bQ(t) = a - b\sum_{j=1} q_j(t).$$

Suppose that a subset of $k$ firms ($2 \leq k \leq n$) engage in rival cross-shareholdings. Following Dai, Benchekroun and Long (2022), we consider a $k$-symmetric cross-ownership structure in which each of the $k$ firms has an equal silent financial stake $v$ in the other firms, while the remaining $n-k$ firms stay independent. We use the subsets $I = \{1, 2, \cdots, k\}$, indexed by $i$ and $O = \{k+1, \cdots, n\}$, indexed by $o$, referring, respectively, to the insiders and outsiders to the cross-ownership. In an industry characterized by symmetric rival cross-shareholdings, the aggregate profits of firm $j$ at time $t$ is:

$$\Pi_j(t) = \pi_j(t) + v\sum_{i\neq j}\Pi_i(t) = p(t)q_j(t) + v\sum_{i\neq j}\Pi_i(t),$$

where $\pi_j(t) = p(t)q_j(t) = (a - b\sum_{j=1} q_j(t))q_j(t)$ denotes firm $j$'s operating profit and $v \in (0, \frac{1}{k-1})$ represents firm $j$'s fractional shareholdings in firm $i$ for any $i \neq j$. [6]

Let $\mathbf{\Pi}$ and $\boldsymbol{q}$ denote the $n \times 1$ vectors of aggregate profits and outputs at time $t$, and $\boldsymbol{D}$ denote the $n \times n$ cross-shareholding matrix, then the aggregate profit functions can be expressed in matrix form as

$$\mathbf{\Pi} = p\boldsymbol{q} + \boldsymbol{D}\mathbf{\Pi},$$

where $\boldsymbol{D} = \begin{bmatrix} \boldsymbol{A}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-k} \end{bmatrix}$, and $\boldsymbol{A}_{kk}$ is a $k \times k$ matrix with element 0 in the diagonal and $v$ off-diagonal. This set of $n$ equations implicitly defines the aggregate profit for each firm at time $t$. Then $\boldsymbol{I} - \boldsymbol{D} = \begin{bmatrix} \boldsymbol{B}_{kk} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix}$, where $\boldsymbol{B}_{kk}$ is a $k \times k$ matrix with element 1 in the diagonal and $-v$ off-diagonal, and $\boldsymbol{I}_{n-k}$ denote the $(n-k) \times (n-k)$ identity matrix. The matrix $\boldsymbol{I} - \boldsymbol{D}$ is invertible, which allows us to solve for the aggregate profit functions:

$$\mathbf{\Pi} = (\boldsymbol{I} - \boldsymbol{D})^{-1}p\boldsymbol{q} = \begin{bmatrix} \boldsymbol{B}_{kk}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix} p\boldsymbol{q},$$

where $\boldsymbol{B}_{kk}^{-1}$ is given by the following matrix

$$\Omega \equiv \frac{1}{f(v)} \begin{bmatrix} 1-(k-2)v & v & \cdots & v \\ v & 1-(k-2)v & \cdots & v \\ \vdots & \vdots & \ddots & \vdots \\ v & v & \cdots & 1-(k-2)v \end{bmatrix},$$

[6] The weight given to rivals' profits is bounded from above by $\frac{1}{k-1}$, which guarantees that the aggregate stake of rivals in each cross-ownership participant, $(k-1)v$, is less than 1. See Dai, Benchekroun and Long (2022) for more explanation on this restriction.

with $f(v) = (1+v)(1-(k-1)v) > 0$. Then the aggregate profit function of firm $i \in I$ at time $t$ is

$$\Pi_i(t) = \frac{a - b\sum_{j\neq i} q_j(t) - bq_i(t)}{f(v)} \left[ (1-(k-2)v)q_i(t) + v \sum_{m\in I\setminus i} q_m(t) \right],$$

while for firm $o \in O$, the aggregate profit function at time $t$ is

$$\Pi_o(t) = (a - b\sum_{j\neq o} q_j(t) - bq_o(t))q_o(t).$$

Taking the strategies of its $(J-1)$ rivals as given, each firm $j$ chooses its own decision rule to maximize the discounted sum of the aggregate profits, which consists of its operating profit and the share of profits obtained through ownership interests in other firms, subject to the stock dynamics. For a typical firm $i \in I$,

$$\max_{q_i(t)\geq 0} \int_0^\infty e^{-rt} \left[ \frac{a - b\sum_{j\neq i} q_j - bq_i}{(1+v)(1-(k-1)v)} \left( (1-(k-2)v)q_i + v \sum_{m\in I\setminus i} q_m \right) \right] dt, \tag{1}$$

$$\text{s.t.} \quad \frac{dS}{dt} = F(S) - q_i - \sum_{j\neq i} q_j, \quad S(0) = S_0, \tag{2}$$

while for a typical firm $o \in O$,

$$\max_{q_o(t)\geq 0} \int_0^\infty e^{-rt} \left[ (a - b\sum_{j\neq o} q_j - bq_o)q_o \right] dt, \tag{3}$$

$$\text{s.t.} \quad \frac{dS}{dt} = F(S) - q_o - \sum_{j\neq o} q_j, \quad S(0) = S_0 \tag{4}$$

where $r > 0$ is the discount rate, the same for all firms.

We make the following assumption:

**Assumption 1.** *The intrinsic growth rate satisfies the following condition:*

$$\delta > \delta_0 \equiv max \left\{ \frac{r\left[ \left( k(k-1)v - n(1+v) \right)^2 + (1+v)^2 \right]}{2(1+v)^2}, \frac{a\left[ \left( k(k-1)v - n(1+v) \right)^2 + (1+v)^2 \right]}{bS_y\left( (k+n+1-k^2)v + n + 1 \right)^2} \right\}$$

Assumption 1 implies that $\frac{\delta}{r}$ is strictly bounded from below, which guarantees that there exists a strictly interior stable steady-state stock. Similar imposition of such a lower bound can be also found in Dutta and Sundaram (1993a,b); Dockner and Sorger (1996); Benchekroun (2003, 2008); Colombo and Labrecciosa (2015, 2018). It follows that

$$\delta > \frac{r}{2} \Longleftrightarrow 2\delta - r > 0, \tag{C1}$$

$$\delta > r \Longleftrightarrow \delta - r > 0, \tag{C2}$$

$$(2\delta - r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big) > 0, \tag{C3}$$

and

$$\delta > \delta_0 \equiv max\left\{\frac{r(n^2+1)}{2}, \frac{a(n^2+1)}{bS_y(n+1)^2}\right\}. \tag{C4}$$

*Proof.* See Appendix A. □

We follow much of the existing literature and restrict our attention to stationary Markov strategies, which are feedback decision rules whereby firms condition their exploitation rates of the resource on the current resource stock:

$$q_j = \phi_j(S(t)).$$

We characterize the Markov perfect Nash equilibrium (MPNE) for the above noncooperative differential game. More specifically,

**Proposition 1.** *Let $\phi_i^*$ and $\phi_o^*$ denote the production strategy for firm $i \in I = \{1, 2, \cdots, k\}$ and firm $o \in O = \{k+1, \cdots, n\}$, respectively:*

$$\phi_i^*(S) = \begin{cases} 0 & \text{for } 0 \le S \le S_i^1, \\ \frac{\big(1-(k-2)v\big)a-(1+v)\big(1-(k-1)v\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b} & \text{for } S_i^1 < S \le S_i^2, \\ q_i^v = \frac{\big(1-(k-2)v\big)a}{\big((k+n+1-k^2)v+n+1\big)b} & \text{for } S > S_i^2, \end{cases} \tag{5}$$

$$\phi_o^*(S) = \begin{cases} 0 & \text{for } 0 \le S \le S_o^1, \\ \frac{1+v}{1-(k-2)v}\frac{\big(1-(k-2)v\big)a-(1+v)\big(1-(k-1)v\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b} & \text{for } S_o^1 < S \le S_o^2, , \\ q_o^v = \frac{(1+v)a}{\big((k+n+1-k^2)v+n+1\big)b} & \text{for } S > S_o^2, \end{cases} \tag{6}$$

*where*

$$A = \frac{b(r-2\delta)\big(1-(k-2)v\big)\Big((k+n+1-k^2)v+n+1\Big)^2}{2(1+v)\big(1-(k-1)v\big)\Big((k+n-k^2)v+n\Big)^2} < 0,$$

$$B = \frac{a(2\delta-r)\big(1-(k-2)v\big)\Big[\Big(k(k-1)v-n(1+v)\Big)^2+(1+v)^2\Big]}{2\delta(1+v)\big(1-(k-1)v\big)\Big((k+n-k^2)v+n\Big)^2} > 0,$$

$$S_i^1 = \frac{\big(1-(k-2)v\big)a-(1+v)\big(1-(k-1)v\big)B}{(1+v)\big(1-(k-1)v\big)A} = S_o^1, \quad S_i^2 = -\frac{B}{A} = S_o^2.$$

*Then, the n-tuple vector of closed-loop strategies $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ constitutes a MPNE in a common property renewable resource oligopoly with cross-ownership.*

*Proof.* See Appendix B. □

Since the thresholds of the stocks are the same for the insiders and outsiders, let $S_1 = S_i^1 = S_o^1$ and $S_2 = S_i^2 = S_o^2$ thereafter for ease of exposition. More specifically after simplification, we have

$$S_1 = \frac{a\left[(2\delta - r)(1+v)^2 - r\Big(k(k-1)v - n(1+v)\Big)^2\right]}{b\delta(2\delta - r)\Big((k+n+1-k^2)v + n + 1\Big)^2}, \; S_2 = \frac{a\left[\Big(k(k-1)v - n(1+v)\Big)^2 + (1+v)^2\right]}{b\delta\Big((k+n+1-k^2)v + n + 1\Big)^2}$$

Proposition 1 shows that firms' exploitation strategies depend crucially on the stock level of the productive asset. To visualize these results, we plot the production strategies of a typical insider ($\phi_i^*$) and an outsider ($\phi_o^*$) as a function of the asset stock (S) in Figure 1.

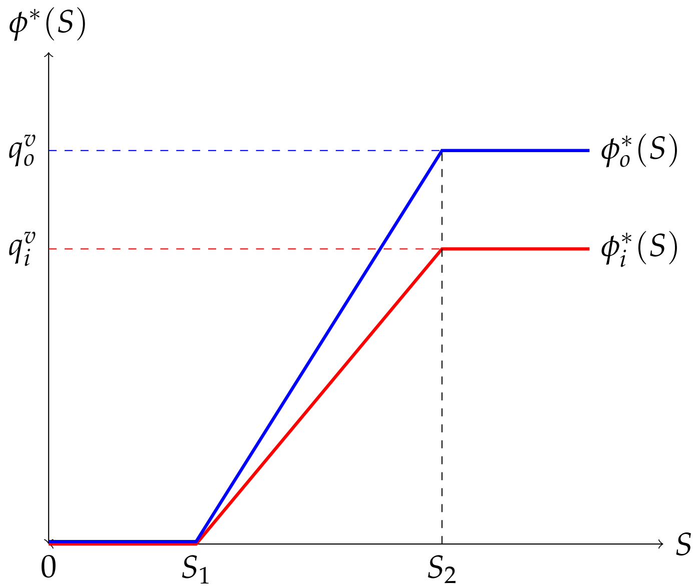



Figure 1: The feedback strategies

When the asset's stock is too small (i.e., $S \leq S_1$), both the insiders and outsiders will voluntarily cease their productions, despite the fact that they have free access to the asset and they compete in the oligopoly market. Similar results can be also found in Benhabib and Radner (1992), Benchekroun (2003, 2008) and Colombo and Labrecciosa (2018) where firms refrain from consumption/production and wait for the asset to grow to the maturity threshold. Therefore, depletion of the asset is avoided. When the level of the stock is very large or abundant (i.e., $S > S_2$), firms simply adopt the production strategy that coincides with the solution under a static Cournot game with a $k$-symmetric cross-ownership structure when the inverse demand is $p = a - bQ$ and the production cost is $c = 0$, i.e., $q_i^v$ and $q_o^v$ in Dai, Benchekroun and Long (2022).

Finally, when the stock level of the asset is intermediate (i.e., $S_1 < S \leq S_2$), both the cross-ownership participants and non-participants will adopt a Markov strategy that is a monotonous non-decreasing function of $S$, strictly increasing over $S_1$ to $S_2$.

Note that we would need the assumption made from Assumption 1 to ensure that

$$S_y > S_2 \Longleftrightarrow \delta > \frac{a\left[\left(k(k-1)v - n(1+v)\right)^2 + (1+v)^2\right]}{bS_y\left((k+n+1-k^2)v + n + 1\right)^2}, \forall\, v \in (0, \frac{1}{k-1})$$

or equivalently,

$$\delta > \frac{a(n^2+1)}{bS_y(n+1)^2}.$$

When $S_y < S_2$ , the N-tuple vector of closed-loop strategies $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ is not a global MPNE.

Let $S^*(t)$ denote the equilibrium time path of the asset and $\Phi_v^*(S)$ denote the industry's production, i.e., $\Phi_v^*(S) = k\phi_i^*(S) + (n-k)\phi_o^*(S)$, or

$$\Phi_v^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_1 \\ \frac{(-k^2+n+k)v+n}{1-(k-2)v} \frac{\left(1-(k-2)v\right)a - (1+v)\left(1-(k-1)v\right)(AS+B)}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S_1 < S \leq S_2 \\ Q_v = \frac{\left((-k^2+n+k)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S > S_2 \end{cases}.$$

Then, we have the following:

**Corollary 1.** *(i) For $\delta S_y < Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$, there exists a unique positive stationary asset stock given by*

$$S_1^\infty = \frac{a\left[(2\delta - r)(1+v)^2 - r\left(k(k-1)v - n(1+v)\right)^2\right]}{b\delta\left((k+n+1-k^2)v+n+1\right)\left[(2\delta-r)(1+v) + r\left(k(k-1)v - n(1+v)\right)\right]} \in (S_1, S_2)$$

*that is globally asymptotically stable with*

$$\lim_{t\to\infty} S^*(t) = S_1^\infty, \quad \textit{for any } S_0 > 0;$$

*(ii) For $\delta S_y > Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$, there exist three positive stationary asset stocks, denoted by $S_1^\infty, S_2^\infty$, and $S_3^\infty$ with*

$$S_1 < S_1^\infty < S_2 < S_2^\infty < S_3^\infty,$$

*where*

$$S_2^\infty = \frac{a\Big((k+n-k^2)v+n\Big)}{b\delta\Big((k+n+1-k^2)v+n+1\Big)}, \quad S_3^\infty = 1 - \frac{a(1-S_y)\Big((k+n-k^2)v+n\Big)}{b\delta S_y\Big((k+n+1-k^2)v+n+1\Big)}.$$

*For any initial stock $S_0 \in (0, S_2^\infty)$, the asset stock converges to $S_1^\infty$, while for any $S_0 \in (S_2^\infty, \infty)$, the asset stock converges to $S_3^\infty$.*

*Proof.* See Appendix C. □

Corollary 1 demonstrates that when the static Cournot industry output with cross-ownership is larger than the maximum sustainable yield as in case (i), there is a unique positive steady-state stock to which the MPNE path of the asset's stock converges. However, if the maximum sustainable yield exceeds the static Cournot industry output with cross-ownership as in case (ii), there exist three positive steady-state stocks with the smallest and largest being stable and the middle one unstable. The above findings in cases (i) and (ii) of Corollary 1 are illustrated in Figure 2 and Figure 3, respectively. Note that when the initial resource stock and the implicit growth rate

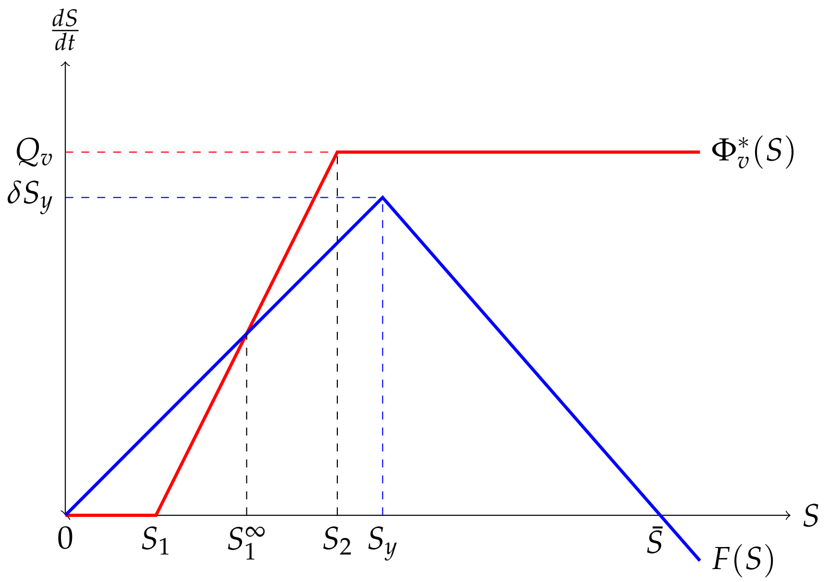



Figure 2: The stationary stock when $\delta S_y < Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$

of the asset are large enough (i.e., $S_0 > S_2^\infty$ and $\delta > \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)bS_y}$), exploiting the asset at a rate corresponding to the static Cournot equilibrium with cross-ownership $(q_i^v, \cdots, q_i^v, q_o^v, \cdots, q_o^v)$ is sustainable as a MPNE. Firms can play this equilibrium endlessly. Interestingly, the steady state level of the asset stock $S_3^\infty$ in this case does not depend on the discount rate $r$, which implies that the resource dynamics plays no role. However, when the implicit growth rate of the asset is not high

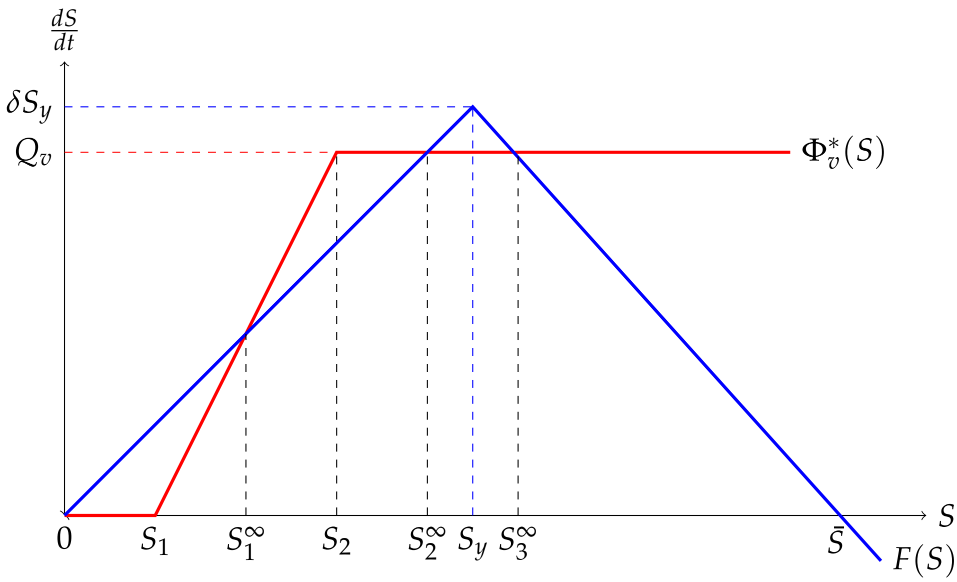



Figure 3: The stationary stocks when $\delta S_y > Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$

enough (i.e., $\delta < \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)bS_y}$), playing the static Cournot equilibrium with cross-ownership is not sustainable. While firms can still adopt such strategies if the initial asset stock is above $S_2^\infty$, these production rates will only last for a finite period of time, after which firms will reduce their production as the stock level of the asset falls below $S_2^\infty$ and eventually converges to $S_1^\infty \in (S_1, S_2)$.

# 3 The profitability of cross-ownership in a renewable resource industry

In this section, we exploit the characterization of the MPNE of the above-defined game and examine the profitability of cross-ownership in the context of a common property renewable resource oligopoly. Following Dai, Benchekroun and Long (2022), we define the profitability of cross-ownership as the difference between the equilibrium discounted sum of operating profits with and without cross-ownership. We then compare it with the static case and show that the cross-ownership paradox result may not hold in a renewable resource industry. Specifically, we show both analytically and numerically that there always exists an interval of the renewable resource stock for which a $k$-symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are unprofitable in the corresponding static equilibrium framework.

It should be noted that throughout this section, the analysis is conducted in the short run, i.e., in the neighbourhood of a given initial resource stock $S_0$. Another crucial assumption we make in what follows is that rival cross-shareholdings occur only at time $t = 0$ with $S_0 > 0$ and these shareholdings are irreversible. This is an important consideration in our dynamic framework because the timing of cross-ownership can become an issue as opposed to the purely static case. A symmetric cross-ownership might be profitable for only a finite period of time, after which it would be dissolved. Hence, it suffices to confine our focus to such a situation, as our primary aim is to illustrate circumstances under which the profitability of cross-ownership is positive in our dynamic framework but would not be in the static counterpart.

### 3.1 The effects on production

Before we delve into the detailed profitability analysis, it is useful to compare the equilibrium strategies under cross-ownership with the ones under the standard common property renewable resource oligopoly model without cross-ownership as in Benchekroun (2008). From Proposition 1 and Appendix A in Benchekroun (2008), the benchmark equilibrium feedback strategy for a typical firm without cross-ownership is given by

$$\phi_c^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_{1,N}, \\ \frac{a-(XS+Y)}{(n+1)b} & \text{for } S_{1,N} < S \leq S_{2,N}, \\ q_c = \frac{a}{(n+1)b} & \text{for } S > S_{2,N}, \end{cases}$$

where

$$X = \frac{b(r-2\delta)(1+n)^2}{2n^2} < 0, \quad Y = \frac{a(2\delta-r)(1+n^2)}{2\delta n^2} > 0,$$

$$S_{1,N} = \frac{a-Y}{X} = \frac{a\big(2\delta - r(1+n^2)\big)}{b\delta(2\delta-r)(1+n)^2}, \quad S_{2,N} = -\frac{Y}{X} = \frac{a(1+n^2)}{b\delta(1+n)^2}.$$

It can be easily observed that when $v=0$, $\phi_i^*(S) = \phi_o^*(S) = \phi_c^*(S)$. Let $\Omega_i$ and $\Omega_o$ denote the slopes of the equilibrium feedback strategies of a typical insider firm $\phi_i^*(S)$ and an outsider firm $\phi_o^*(S)$ under cross-ownership when $S \in [S_1, S_2]$, and denote by $\Omega_c$ the positive slope of the benchmark equilibrium strategy $\phi_c^*(S)$ when $S \in [S_{1,N}, S_{2,N}]$, respectively:

$$\Omega_i = \frac{(2\delta-r)\big(1-(k-2)v\big)\big((k+n+1-k^2)v+n+1\big)}{2\big((k+n-k^2)v+n\big)^2},$$

$$\Omega_o = \frac{(2\delta-r)(1+v)\big((k+n+1-k^2)v+n+1\big)}{2\big((k+n-k^2)v+n\big)^2}, \quad \Omega_c = \frac{(2\delta-r)(1+n)}{2n^2}.$$

We distinguish between the case of $n=k$ in which all the firms in the industry engage in rival cross-shareholdings and the case of $n>k$ where only a subset of firms hold shares in each other.[7] Then, we have the following results:

**Lemma 1.** *For any* $0 < v < \frac{1}{k-1}$ *and* $n = k \geq 2$, $S_1 > S_{1,N}, S_2 < S_{2,N}, q_c > q_i^v$, *and* $\Omega_i > \Omega_c$.

[7] When $n=k$, the set $O$ is empty. Thus by definition, $\phi_o^*$ does not exist as there are no outsiders. So we only have $\phi_i^*$ in the case of $n$-symmetric cross-ownership, as if firms were merging.

**Lemma 2.** *For any* $0 < v < \frac{1}{k-1}$ *and* $n > k \geq 2$, $S_1 > S_{1,N}$, $S_2 < S_{2,N}$, $q_i^v < q_c < q_o^v$, *and* $\Omega_i < \Omega_c < \Omega_o$.

*Proof.* See Appendix D. □

Lemma 1 and 2 show that when competing firms in the same industry hold shares in each other, it has the following impact: (i) the range of asset stocks for which firms adopt the positive and increasing Markov strategies ($[S_1, S_2]$) shrinks due to the fact that the maturity threshold ($S_1$) increases while the threshold beyond which firms commit to the static Cournot strategies ($S_2$) decreases; (ii) for $S \in [S_1, S_2]$ and any $n = k$, i.e., when all firms engage in cross-ownership, the linear curve of each firm becomes steeper; (iii) for $S \in [S_1, S_2]$ and any $n > k$, the outsiders become more responsive to changes in the asset's stock while the insiders become less responsive, i.e., the linear curve of a typical outsider becomes steeper while that of an insider becomes flatter; (iv) the result that $q_i^v$ decreases and $q_o^v$ increases directly follows from the static theory: when firms hold an ownership stake in their competitors, they will compete less aggressively and thus reduce their outputs, because their profit gains may come at the loss of other firms in which they have shareholdings. But in terms of strategic substitutes in Cournot competition, outsiders will respond by increasing their production.

Further, we can establish the following proposition that compares the individual equilibrium production strategies with and without cross-ownership under $n = k$.

**Proposition 2.** *For any* $n = k \geq 3$ *and* $v \in (0, \hat{v})$, *there exists a* $\hat{S}_1 \in (S_1, S_2)$ *and a* $\hat{S}_2 \in (S_2, S_{2,N})$ *such that* $\phi_i^*(S) > \phi_c^*(S)$ *if and only if* $\hat{S}_1 < S < \hat{S}_2$, *where*

$$\hat{v} = \frac{(n+1)\Big(r - 2\delta + n(\delta - r)\Big)}{(\delta - r)\big(n^2(n-2)\big) + 2\delta - r + n\delta};$$

*otherwise,* $\phi_c^*(S) \geq \phi_i^*(S)$ *for all other* $S$. *Moreover,* $\phi_c^*(S) \geq \phi_i^*(S)$ *for any* $S$ *if any of the following conditions is satisfied:*

$$(i)\ n = k = 2 \quad and \quad v \in (0, \frac{1}{n-1});$$

*or*

$$(ii)\ n = k \geq 3, \quad and \quad v \in [\hat{v}, \frac{1}{n-1}).$$

*Proof.* See Appendix E. □

The findings of Proposition 2 are two-fold, which can be illustrated in Figure 4 that plots the possible scenarios for the individual production strategies in the cases with and without cross-ownership when all the firms in the industry engage in rival cross-shareholdings. First, it demonstrates that for any given resource stock such that exploitation rates are strictly positive (i.e., $S > S_{1,N}$), interlock cross-ownership results

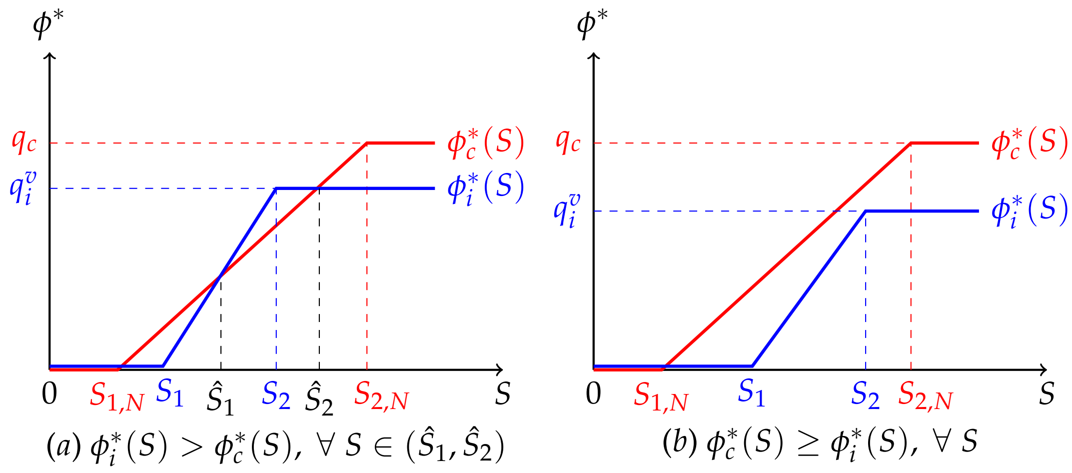



Figure 4: Individual MPNE with and without cross-ownership when $n = k$

in a lower production for each individual firm (i.e., $\phi_i^* < \phi_c^*$), either when a duopoly controls the industry or when the number of firms in the industry is large but the ownership stake is high enough, as shown in Figure 4(b). This result is consistent with the traditional oligopoly theory, as when firms cooperate due to their ownership stakes, each firm will unilaterally reduce its output to lessen competition. Colombo and Labrecciosa (2018) also find similar results in a duopoly setting. However, our result extends beyond their 2-firm case and holds in a more general oligopoly setting for a large enough ownership stake.

Second, Proposition 2 also states that under some conditions, there exists an interval of initial resource stocks for which the cross-owner participants may increase their production following the symmetric cross-ownership, as shown in Figure 4(a). This result is quite surprising and goes against the static oligopoly theory and cross-ownership theory. Indeed, when all firms participate in cross-ownership, they have an incentive to reduce competition as one firm's gain may come at the loss of the other firms it has shareholdings. As such, every cross-ownership participant unilaterally reduces its output in terms of Cournot competition. However, in our context where an oligopoly exploits a common property productive asset, there is another channel through which cross-ownership affects firms' extraction rate, in addition to the reduced competition at the level of the output market. That is, cross-ownership also affects how firms interact with each other at the resource level.

To explain this further, note that the value function of a typical insider firm $i \in I$, which corresponds to the sum of the aggregate profits along the equilibrium path discounted to infinity at rate $r$ when the asset's stock is $S$, is denoted by (See Proposition 1 and Appendix B)

$$V_i(S) = \begin{cases} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ W(S) = \frac{A}{2}S^2 + BS + C & \text{for } S_1 < S \leq S_2 \\ \frac{\Pi_i}{r} & \text{for } S > S_2 \end{cases} .$$

For $S > S_2$, the value function of a typical insider firm $i \in I$ is equal to the aggregate profit in the static Cournot equilibrium with cross-ownership discounted to infinity at $r$, independent of the asset's stock. This means that for any $S > S_2$, resource scarcity plays no role and the value of an insider firm is simply the discounted market rent due to the induced market power by cross-ownership. On the other hand for any $S < S_2$, the value function accounts for both the market rent and the resource rent, which depends on the stock of the asset. More specifically, the value to an insider firm $i \in I$ of an additional until of common stock $S$, or the resource rent, can be determined as

$$\frac{\partial V_i(S)}{\partial S} = V_i'(S) = \begin{cases} \frac{r}{\delta S_1}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}-1} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ W'(S) = AS + B & \text{for } S_1 < S \leq S_2 \\ 0 & \text{for } S > S_2 \end{cases}.$$

It can be seen from the above that for $S < S_2$, the rent of an additional unit of stock to the insider is decreasing with the stock of the asset, and it tends to infinity as the stock approaches zero.

In the absence of resource rents, reduced competition in the output market due to cross-ownership leads to a decrease in the production of each firm. However, in the presence of resource scarcity, i.e., when the stock of the asset is below the threshold $S_2$ beyond which firms commit to the static Cournot strategies, the resource rent is positive and affects firms' production. The impact of rival cross-shareholdings on the resource rent manifests as a negative effect for relatively large stocks and a positive effect for relatively small stocks.

Notice that this specific outcome – the insiders may increase their production as a result of cross-ownership – occurs only when the number of firms is larger than two (i.e., $n = k \geq 3$) and the ownership stake is not large enough (i.e., $v \in (0, \hat{v})$. That is, cross-owners will only slightly reduce their production due to a relatively low shareholding in the output market, but this results in a sufficiently abundant stock of the asset, which in turn reduces the marginal valuation of the resource stock by each cross-ownership participant. A smaller resource rent thus incentivizes each cross-owner to expand its production. This output expansion is the resource rent effect of cross-ownership. The first part of Proposition 2 and Figure 4(a) show that the effect on individual firm's production of the change in the resource rent due to cross-shareholdings outweighs the effect of reduced competition at the output market for $n = k \geq 3$, $v \in (0, \hat{v})$ and $S \in (\hat{S}_1, \hat{S}_2)$.

Now we turn to the case of $n > k$ and we can assert the following:

**Proposition 3.** *For any $n > k \geq 2$ and $0 < v < \frac{1}{k-1}$, there exists a $\tilde{S} \in (S_1, S_2)$ such that*

$$\phi_c^*(S) - \phi_o^*(S) \begin{cases} > 0 \;\; \text{if} \;\; S_{1,N} < S < \tilde{S} \\ \leq 0 \;\; \text{if} \;\; S \geq \tilde{S} \end{cases}$$

*while*

$$\phi_c^*(S) > \phi_i^*(S) \quad \textit{for } S > S_{1,N},$$

*and*

$$\phi_i^*(S) = \phi_c^*(S) = \phi_o^*(S) = 0 \quad \textit{for } S \leq S_{1,N}.$$

*Proof.* Directly follow from Lemma 2. □

Figure 5 illustrates the findings of Proposition 3. Except for the region $S \leq S_{1,N}$

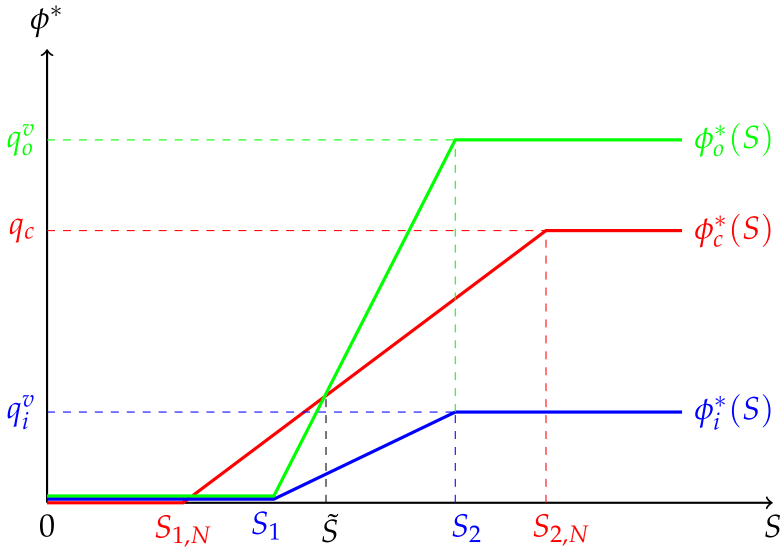



Figure 5: Individual MPNE with and without cross-ownership when $n > k$

where both the insiders and outsiders keep their production at zero, the production of a typical insider is strictly lower than the one without cross-ownership, while the production of a typical outsider is smaller for low stocks ($S_{1,N} < S < \tilde{S}$) and larger for high stocks ($S > \tilde{S}$) than the case without cross-ownership. That is, there exists a range of initial resource stocks such that the outsiders may also lower their production as a result of cross-ownership. This result is quite counterintuitive, as one would expect that in terms of strategic substitutes in Cournot competition, outsiders will always expand their production in response to the output reduction brought by cross-owners.

However, in our dynamic case where firms exploit a common stock while cross-ownership non-participants exhibit a more pronounced response to the asset's stock compared to the participants, decreasing production is also in the best interest of these outsiders for the relatively low levels of stocks ($S_{1,N} < S < \tilde{S}$). In particular, the outsiders will voluntarily cease their production for $S \leq S_1$ and leave the asset to grow. As the stock gradually increases within the range $(S_1, \tilde{S})$, the asset remains sufficiently scarce, leading to an increase in the marginal valuation of the resource stock for each cross-ownership non-participant. A higher resource rent thus provides an incentive for the outsiders to reduce their production. Proposition 3 and Figure 5 show that the output reduction resulting from the resource rent effect of cross-ownership dominates the static effect of production expansion for $S_{1,N} < S < \tilde{S}$.

Next, we compare the equilibrium industry outputs with and without cross-ownership. Recall that the total industry production with cross-ownership is denoted by

$$\Phi_v^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_1 \\ \frac{(-k^2+n+k)v+n}{1-(k-2)v} \frac{\left(1-(k-2)v\right)a-(1+v)\left(1-(k-1)v\right)(AS+B)}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S_1 < S \leq S_2 \\ Q_v = \frac{\left((-k^2+n+k)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S > S_2 \end{cases},$$

while the one without cross-ownership is given by

$$\Phi_c^*(S) = n\phi_c^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_{1,N} \\ \frac{n\left(a-(XS+Y)\right)}{(n+1)b} & \text{for } S_{1,N} < S \leq S_{2,N} \\ Q_c = \frac{na}{(n+1)b} & \text{for } S > S_{2,N} \end{cases}.$$

Following Proposition 2, an immediate result can be established regarding the comparison of the industry equilibrium production strategies with and without cross-ownership when $n = k$. We summarize it in Corollary 2 and illustrate it in Figure 6 below.

**Corollary 2.** *For any $n = k \geq 3$ and $v \in (0, \hat{v})$, there exists a $\hat{S}_1 \in (S_1, S_2)$ and a $\hat{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ if and only if $\hat{S}_1 < S < \hat{S}_2$, where*

$$\hat{v} = \frac{(n+1)\Big(r - 2\delta + n(\delta - r)\Big)}{(\delta - r)\big(n^2(n-2)\big) + 2\delta - r + n\delta};$$

*otherwise, $\Phi_c^*(S) \geq \Phi_v^*(S)$ for all other $S$. Moreover, $\Phi_c^*(S) \geq \Phi_v^*(S)$ for any $S$ if any of the following conditions is satisfied: (i) $n = k = 2$, or (ii) $n = k \geq 3$ and $v \in [\hat{v}, \frac{1}{n-1})$.*

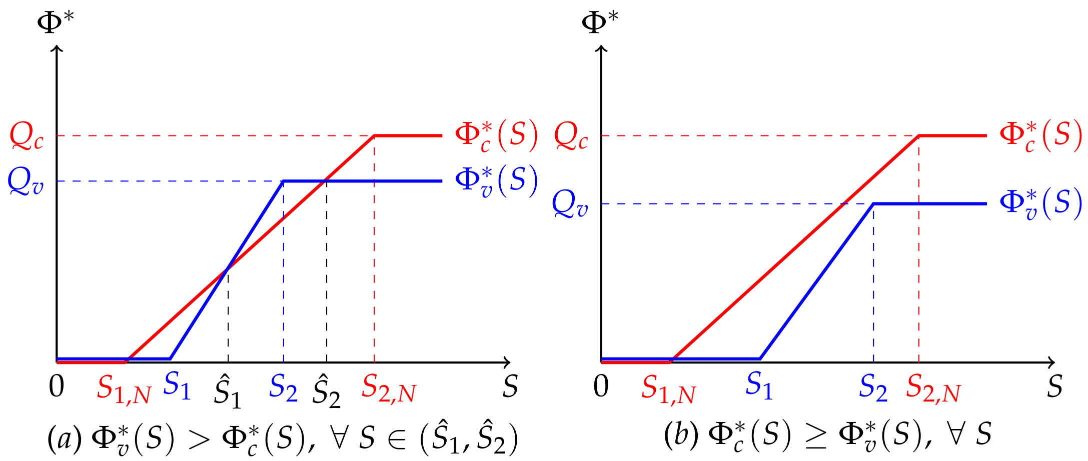



Figure 6: Industry MPNE with and without cross-ownership when $n = k$

Now, let us turn to the case of $n > k$. Denote by $\zeta_v$ and $\zeta_c$ the positive slopes of the industry production for the cases with and without cross-ownership, respectively:

$$\zeta_v = \frac{(2\delta - r)\Big((k+n+1-k^2)v + n + 1\Big)}{2\Big((k+n-k^2)v + n\Big)}, \quad \zeta_c = \frac{(2\delta - r)(1+n)}{2n}.$$

then we can easily observe the following result:

**Lemma 3.** *For any $n \geq k \geq 2$ and $0 < v < \frac{1}{k-1}$, $Q_v < Q_c$ and $\zeta_v > \zeta_c$.*

*Proof.* See Appendix F. □

Further, we can establish the following proposition that compares the industry equilibrium production strategies with and without cross-ownership in the case of $n > k$.

**Proposition 4.** *For any $n > k \geq 2$ and $v \in (0, \frac{1}{k-1})$, there exists a $\tilde{S}_1 \in (S_1, S_2)$ and a $\tilde{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ if and only if $S \in (\tilde{S}_1, \tilde{S}_2)$.*

*Proof.* See Appendix G. □

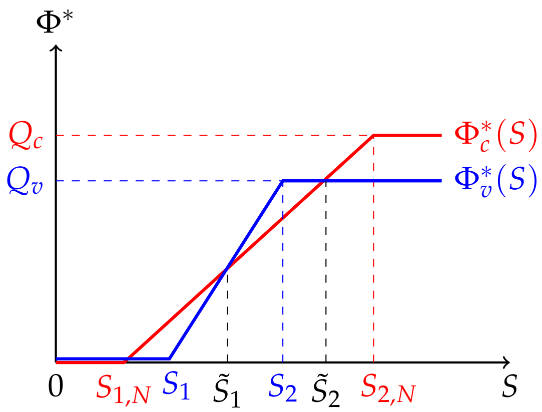



Figure 7: Industry MPNE with and without cross-ownership when $n > k$

Similar to Corollary 2, Proposition 4 also demonstrates that there exists some range of resource stocks such that the total industry production can increase as a result of cross-ownership. This surprising result, illustrated in Figure 7, contradicts the traditional oligopoly and cross-ownership theory. Indeed, when firms acquire an ownership stake in their rivals, they compete less aggressively and thus unilaterally reduce their output. But in terms of strategic substitutes in Cournot competition, firms that do not participate in cross-ownership will respond by expanding their production. However, the output reduction brought by cross-owners will outweigh the outsiders' production expansion, leading to an overall decrease in industry production. This is the standard static result of output reduction brought by cross-ownership. But as explained earlier, in our dynamic context where an oligopoly exploits a common property renewable resource, the output expansion resulting from the resource rent effect due to the change in the marginal valuation of the resource stock by firms as a result of cross-ownership will dominate the static effect for any $S \in (\tilde{S}_1, \tilde{S}_2)$. Outside this range, the industry output is strictly lower following cross-ownership, except for $S \leq S_{1,N}$ where the production remains at zero.

Despite these solid findings, one may question whether those situations could ever occur, as firms may never find it profitable to engage in rival cross-shareholdings in the first place. We thus move to conduct the profitability analysis of cross-ownership in a renewable resource industry in the next subsection and compare it to the static case.

## 3.2 The profitability of cross-ownership

In a static framework, three countervailing effects are in operation when firms decide to participate in cross-ownership in an oligopolistic market. One is the positive effect on cross-owners' profits due to the partial elimination of previous rivalry, the second is the negative effect of non-participants' production expansion in terms of strategic substitutability, and the last one is how aggressively outsiders will respond depending on the levels of shareholdings. The relative size of these three effects drives the final result concerning the profitability of cross-ownership. More specifically, firms can never profit from cross-shareholdings if $\frac{k}{n} \leq \frac{k}{2k-1}$, but will also have an incentive to do so if $\frac{k}{n} > \frac{k}{k+\sqrt{k}-1}$; for participation ratios in between the lower threshold ($\frac{k}{2k-1}$) and upper threshold ($\frac{k}{k+\sqrt{k}-1}$), there exists a large range of shareholdings for which a $k$-symmetric cross-ownership can be profitable (See Proposition 2 in Dai, Benchekroun and Long (2022)).

However, as we shall show below, this static result may not necessarily carry over to the case of a common property renewable resource industry where the resource stock, if left unexploited, reproduces itself naturally at a rate that depends on the size of the stock. To see this, note that the equilibrium discounted sum of operating profits for a typical insider firm $i \in I$ under the $k-$symmetric cross-ownership structure is given by [8]

$$V_S = (1-(k-1)v)V_i(S) = \begin{cases} (1-(k-1)v)\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ (1-(k-1)v)W(S) & \text{for } S_1 < S \leq S_2 \\ \frac{\pi_i^v}{r} & \text{for } S > S_2 \end{cases},$$

where

$$W(S) = \frac{A}{2}S^2 + BS + C, \quad \pi_i^v = \frac{a^2(1+v)\left(1-(k-2)v\right)}{b\left((k+n+1-k^2)v+n+1\right)^2},$$

[8] Recall that $V_i(S)$ corresponds to the equilibrium discounted sum of aggregate profits or accounting profits for a typical insider firm, which includes not only the profits from its own operations but also the share of profits in other firms. We use the operating profits or economic profits rather than the accounting profits to define profitability.

while that for an individual firm without cross-ownership (as shown in Benchekroun (2008) Proposition 1 and Appendix A) is given by

$$V_c = \begin{cases} \left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S \leq S_{1,N} \\ W_c(S) & \text{for } S_{1,N} < S \leq S_{2,N} \\ \frac{\pi_c}{r} & \text{for } S > S_{2,N} \end{cases},$$

where

$$W_c(S) = \frac{X}{2}S^2 + YS + Z, \quad \pi_c = \frac{a^2}{b(n+1)^2},$$

$$X = \frac{b(r-2\delta)(1+n)^2}{2n^2} < 0, \quad Y = \frac{a(2\delta - r)(1+n^2)}{2\delta n^2} > 0,$$

$$Z = \frac{a^2\left(2\delta - r(1+n^2)\right)\left(2\delta n^2 - r(1+n^2)\right)}{4br\delta^2 n^2(1+n)^2},$$

$$S_{1,N} = \frac{a-Y}{X} = \frac{a\left(2\delta - r(1+n^2)\right)}{b\delta(2\delta - r)(1+n)^2}, \quad S_{2,N} = -\frac{Y}{X} = \frac{a(1+n^2)}{b\delta(1+n)^2}.$$

Then, a $k-$symmetric cross-ownership is profitable when

$$G(k,n,v,S) = V_S - V_c > 0.$$

Clearly, the profitability of cross-ownership in a common pool renewable resource industry depends on $k, n, v$, but also crucially depends on the asset stock $S$.

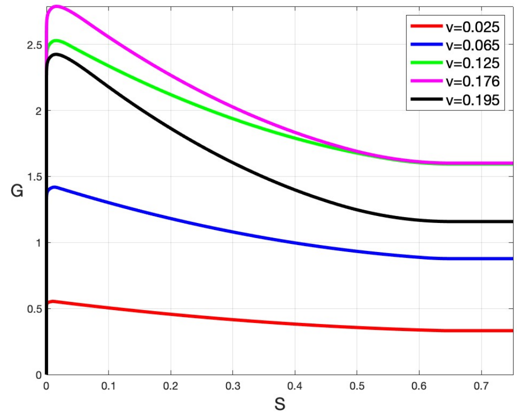
It is useful to distinguish five regions for the resource stocks, with Region I: $S \in [0, S_{1,N})$, Region II: $S \in [S_{1,N}, S_1)$, Region III: $S \in [S_1, S_2)$, Region IV: $S \in [S_2, S_{2,N})$, and Region V: $S \in [S_{2,N}, \infty)$. The profitability function $G(k,n,v,S)$ can then be expressed as

$$G(k,n,v,S) = \begin{cases} (1-(k-1)v)\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - \left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S < S_{1,N} \\ (1-(k-1)v)\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - W_c(S) & \text{for } S_{1,N} \leq S < S_1 \\ (1-(k-1)v)W(S) - W_c(S) & \text{for } S_1 \leq S < S_2 \\ \frac{\pi_i^v}{r} - W_c(S) & \text{for } S_2 \leq S < S_{2,N} \\ \frac{\pi_i^v}{r} - \frac{\pi_c}{r} & \text{for } S \geq S_{2,N} \end{cases}.$$

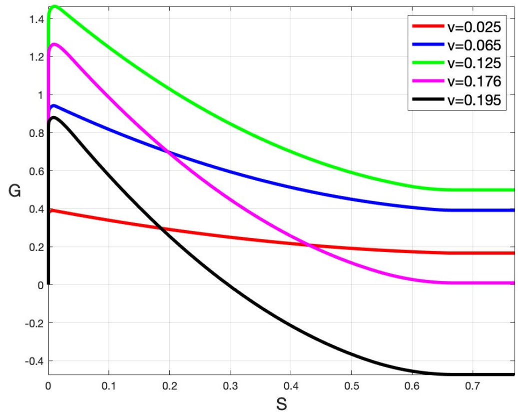
It can be easily observed that for any $S \geq S_{2,N}$, the dynamic profitability is simply equal to the static profitability discounted to infinity at rate $r$. That is, the cross-ownership paradox result obtained in Dai, Benchekroun and Long (2022) will also hold in the dynamic equilibrium for any $S$ that is large enough. However, our dynamic profitability holds beyond this static result. More specifically, we have

**Result 1.** *If a k-symmetric cross-ownership is profitable in the static Cournot equilibrium, it will also be profitable in the dynamic equilibrium for all $S$.*

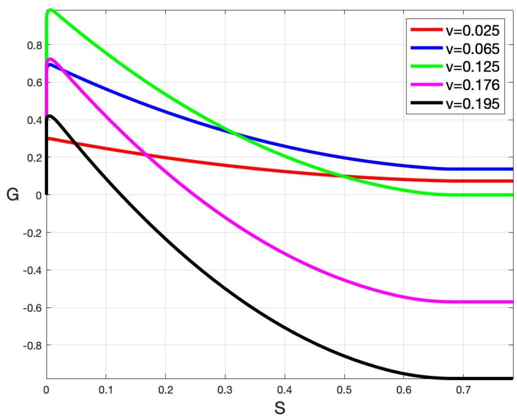
To illustrate this result, we retain the numerical examples used in Dai, Benchekroun and Long (2022) and plot in Figure 8 the dynamic profitability as a function of the resource stock when the participation ratio satisfies $\frac{k}{n} > \frac{k}{2k-1}$. Using parameter values of $a = 5, b = 0.5, r = 0.15, S_y = 0.75, \delta = 12$, we fix $k = 6$ and vary $n = 7, 8, 9, 10$ and plot $G$ as a function of $S$ for different levels of ownership $v$. As a direct comparison, we



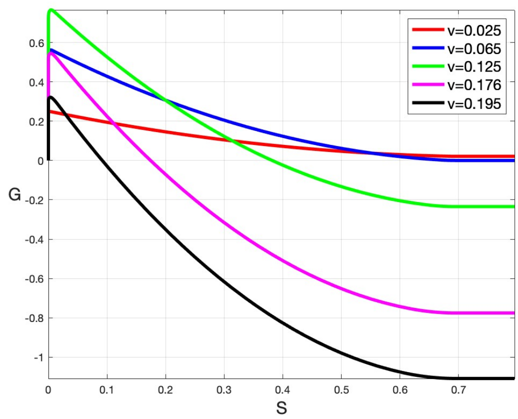

(a) $k = 6, n = 7$




(b) $k = 6, n = 8$




(c) $k = 6, n = 9$



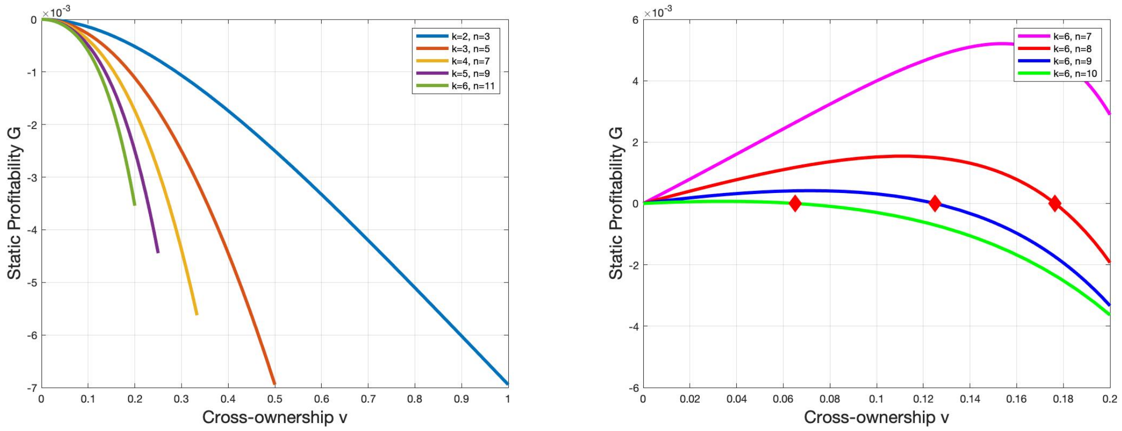

(d) $k = 6, n = 10$

Figure 8: Dynamic profitability as a function of $S$ when $\frac{k}{n} > \frac{k}{2k-1}$

reproduce in Figure 9 the static profitability as a function of the ownership level $v$. As shown in Figure 9(b), when $k$ is fixed at 6, the static profitability of cross-ownership is always positive for any admissible $v \in (0, \frac{1}{k-1})$ when $n = 7$, while for $n = 8, 9, 10$, it is positive if $v < 17.6\%, 12.5\%, 6.5\%$, respectively. Further, we can observe that this static result also carries over to the renewable resource industry for all $S$ as long as $v$ satisfies the corresponding conditions throughout Figures 8(a), 8(b), 8(c) and 8(d). Simulations using many other combinations of $k$ and $n$ satisfying $\frac{k}{n} > \frac{k}{2k-1}$ also find such findings, suggesting that Result 1 is quite robust.

Moreover, a closer look at Figure 8 seems to indicate that there exists a range of initial resource stocks such that the symmetric cross-ownership can be profitable, even though it is unprofitable in the corresponding static framework.[9] For instance, in the

[9]It should be noted that the segment for $G$ when $S$ is small is not a vertical line. This is because the




(a) $\frac{k}{n} \leq \frac{k}{2k-1}$ (b) $\frac{k}{n} > \frac{k}{2k-1}$

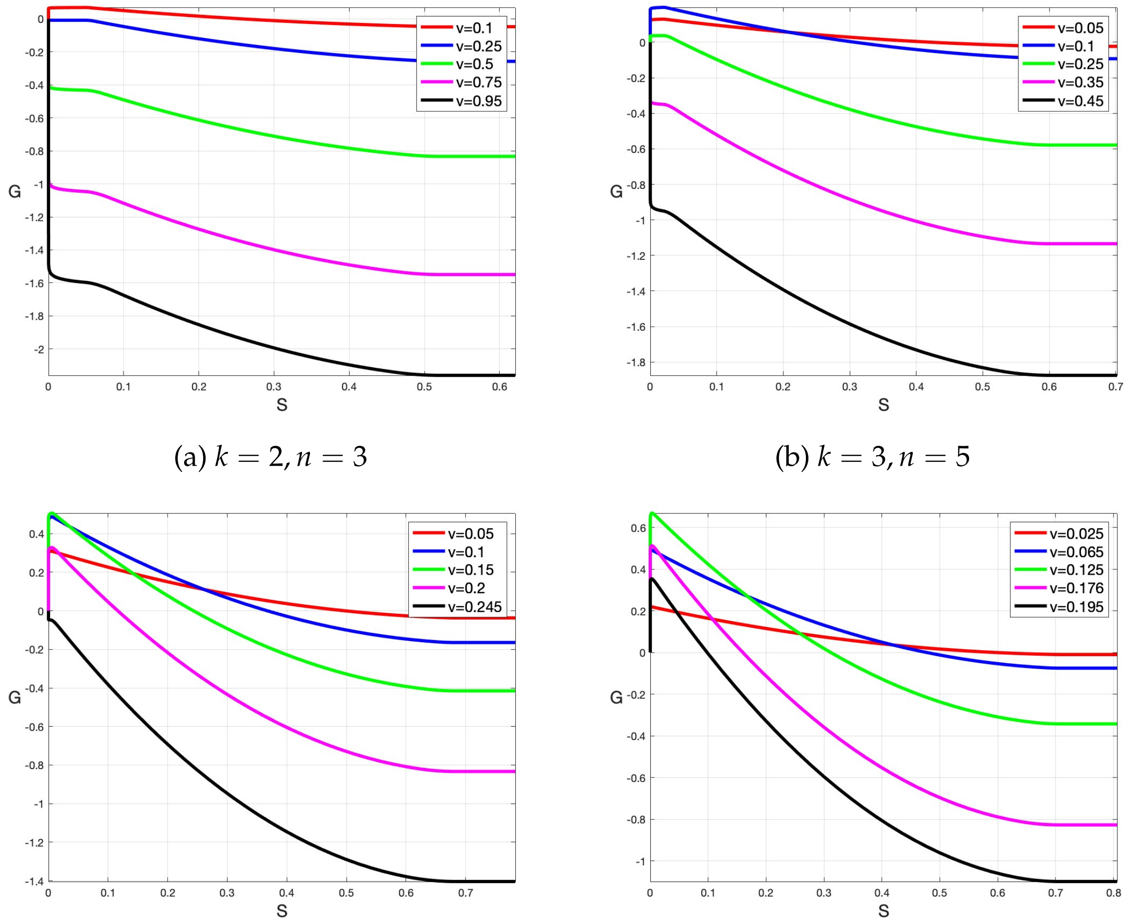
Figure 9: Static profitability as a function of $v$ in Dai, Benchekroun and Long (2022)

case of $k = 6$ and $n = 8$, the static profitability is negative if each of the 6 firms holds more than 17.6%, but as shown in Figure 8(b), the symmetric cross-ownership can be profitable even for $v > 17.6\%$ over some range of resource stocks. Similarly as illustrated in Figure 8(c), the dynamic profitability for $v > 12.5\%$ when $k = 6$ and $n = 9$ is positive over some interval of the initial resource stocks, while such cross-shareholdings are not profitable in the static model. Moreover, compared to the static case where the symmetric cross-ownership is not profitable for $v > 6.5\%$ when $k = 6$ and $n = 10$, it can be profitable in the dynamic model for some initial resource stocks, as demonstrated in Figure 8(d). In addition, the larger the shareholding, the smaller the range of initial resource stocks for which the dynamic profitability is positive.

We now move to check whether these findings can also hold when the participation ratio satisfies $\frac{k}{n} \leq \frac{k}{2k-1}$ in which the static profitability is strictly negative for any admissible $v \in (0, \frac{1}{k-1})$ as shown in Figure 9(a). Using the same parameter values as in Figure 8, Figure 10 illustrates the dynamic profitability $G$ as a function of the initial resource stock $S$ for different levels of shareholdings $v$ when $\frac{k}{n} \leq \frac{k}{2k-1}$. While Figures 10(a), 10(b), 10(c) show that for some large level of ownership $v$, firms can never profit from rival cross-shareholdings for all $S$, we also observe that there always exists some range of initial stocks such that the profitability of cross-ownership is positive, even in the least possible case of $k = 2$ and $n = 3$. In addition, the range of resource stocks for which a $k$-symmetric cross-ownership is profitable shrinks as $v$ increases. Simulations using a wide range of $k$ and $n$ that satisfy $\frac{k}{n} \leq \frac{k}{2k-1}$ also support these findings. We now formally summarize these results in below:

**Result 2.** *There exists an interval of resource stocks such that a k-symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are unprofitable in the corresponding static equilibrium framework. Moreover, this interval decreases in the level of shareholdings.*

values of $S_{1,N}$ and $S_1$ are relatively too small compared to $S_2$ and $S_{2,N}$. Trying to plot the whole region will result in such a display.




(a) $k = 2, n = 3$

(b) $k = 3, n = 5$

(c) $k = 5, n = 9$

(d) $k = 6, n = 11$

Figure 10: Dynamic profitability as a function of $S$ when $\frac{k}{n} \leq \frac{k}{2k-1}$

Up until now, we have not yet explained the difference in the profitability of cross-ownership between the static and dynamic frameworks. However, as already mentioned earlier, a key factor that drives the significantly different results is the presence of a scarcity rent on the renewable resource, which is absent in the static model. Note that from the first-order conditions of the Hamilton-Jacobi-Bellman (HJB) equations to the problems of (1)–(4), the best response functions for a typical insider firm $i \in I$ and an outsider firm $o \in O$ are respectively given by [10]

$$\phi_i = \frac{\big(1-(k-2)v\big)a - \big(1-(k-2)v\big)(n-k)b\phi_o - (1+v)\big(1-(k-1)v\big)V_i'(S)}{\Big[1+k+\big(1-k(k-2)\big)v\Big]b}, \quad (7)$$

and

$$\phi_o = \frac{a - bk\phi_i - V_o'(S)}{b(n-k+1)}, \quad (8)$$

where $V_i'(S) = \frac{\partial V_i(S)}{\partial S}$ and $V_o'(S) = \frac{\partial V_o(S)}{\partial S}$ denote the resource rent or the marginal valuation of an additional unit of the resource stock by cross-ownership participants

[10]Please refer to Appendix B for more details.

and non-participants, respectively. In the purely static Cournot model with cross-ownership, both the terms $V_i'(S)$ and $V_o'(S)$ are 0, as by definition firms' production decisions are independent of $S$. The corresponding pair of reaction functions thus become

$$\phi_i = \frac{(1-(k-2)v)a - (1-(k-2)v)(n-k)b\phi_o}{\Big[1+k+(1-k(k-2))v\Big]b}, \quad \phi_o = \frac{a-bk\phi_i}{b(n-k+1)},$$

in which the best response of an outsider firm to the change in the production of an insider firm due to cross-ownership, or vice versa, is simply the movement along the reaction function. However, in our dynamic framework with the presence of resource rents ($V_i'(S) > 0, V_o'(S) > 0$) as shown in (7) and (8), attempting to move along the reaction functions due to cross-shareholdings will unexpectedly lead to a shift in those reaction functions, because there exists a dynamic link between the level of stock to the rate of production through the growth function in (2) and (4). This additional feature is highly relevant to the profitability analysis of cross-ownership in a renewable resource industry.

To see how these reaction functions are shifting, consider the impact of cross-ownership on the resource rents over the interval $(S_1, S_2)$. Recall that for $S_1 < S < S_2$,

$$V_i(S) = \frac{1}{2}AS^2 + BS + C,$$

$$V_o(S) = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}V_i(S) = \frac{1}{2}DS^2 + ES + F,$$

so the marginal valuation of an additional $S$ for a typical insider firm $i \in I$ is given by

$$V_i'(S) = \frac{\partial V_i(S)}{\partial S} = AS + B,$$

while that for a typical outsider firm $o \in O$ is

$$V_o'(S) = \frac{\partial V_o(S)}{\partial S} = DS + E.$$

Differentiating the resource rents with respect to $v$ yield

$$\frac{\partial V_i'(S)}{\partial v} = \frac{\partial A}{\partial v}\Big(S - S_{iR}\Big) = -\frac{b(k-1)(2\delta - r)\Big((k(k-1)v-(n+1)(1+v)\Big)\Theta_1}{2(1+v)^2\Big(1-(k-1)v\Big)^2\Big((k+n-k^2)v+n\Big)^3}\Big(S - S_{iR}\Big), \tag{9}$$

$$\frac{\partial V_o'(S)}{\partial v} = \frac{\partial D}{\partial v}\left(S - S_{oR}\right) = -\frac{bk(k-1)(2\delta - r)\left((k+n+1-k^2)v+n+1\right)}{\left((k+n-k^2)v+n\right)^3}\left(S - S_{oR}\right), \tag{10}$$

where

$$\begin{aligned}
\Theta_1 =&(k^5 - 4k^4 - 2k^3n + 4k^3 + 6k^2n + k^2 + kn^2 - 3kn - 2k - 2n^2 - 2n)v^4 \\
&+ (-2k^4 - 2k^3n + k^3 + 10k^2n + 9k^2 + 2kn^2 - 6kn - 8k - 6n^2 - 6n)v^3 \\
&+ (-2k^3 + 4k^2n + 12k^2 + kn^2 - 3kn - 12k - 6n^2 - 6n)v^2 \\
&+ (4k^2 - 8k - 2n^2 - 2n)v - 2k,
\end{aligned}$$

$$S_{oR} = -\frac{\frac{\partial E}{\partial v}}{\frac{\partial D}{\partial v}} = \frac{a(1+v)}{b\delta\left((k+n+1-k^2)v+n+1\right)} = \frac{q_o^v}{\delta}, \quad S_{iR} = -\frac{\frac{\partial B}{\partial v}}{\frac{\partial A}{\partial v}}.$$

It would be ideal to explicitly demonstrate how the resource rents are changing with $v$, but as shown in (9), the equation is rather cumbersome. To exemplify the concept, we consider without loss of generality the case of $k = 2$ and $n = 3$, which is supposedly the least likely to be profitable.[11] Then equations (9) and (10) become

$$\frac{\partial V_i'(S)}{\partial v} = \frac{-4b(2\delta - r)(v+2)(v^3+4v^2+6v+1)}{(1-v)^2(1+v)^2(v+3)^3}\left(S - S_{iR}\right),$$

$$\frac{\partial V_o'(S)}{\partial v} = \frac{-4b(2\delta - r)(v+2)}{(v+3)^3}\left(S - S_{oR}\right)$$

where

$$S_{iR} = \frac{a(v^4+6v^3+16v^2+16v+1)}{2b\delta(v+2)(v^3+4v^2+6v+1)}, \quad S_{oR} = \frac{a(1+v)}{2b\delta(v+2)}.$$

It can be easily checked that for any $v \in (0, \frac{1}{k-1})$,

$$S_1 < S_{oR} < S_{iR} < S_2.$$

Therefore, we have

$$\frac{\partial V_i'(S)}{\partial v}\begin{cases} > 0 & \text{for } S_1 < S < S_{iR} \\ = 0 & \text{for } S = S_{iR} \\ < 0 & \text{for } S_{iR} < S < S_2 \end{cases}, \tag{11}$$

and

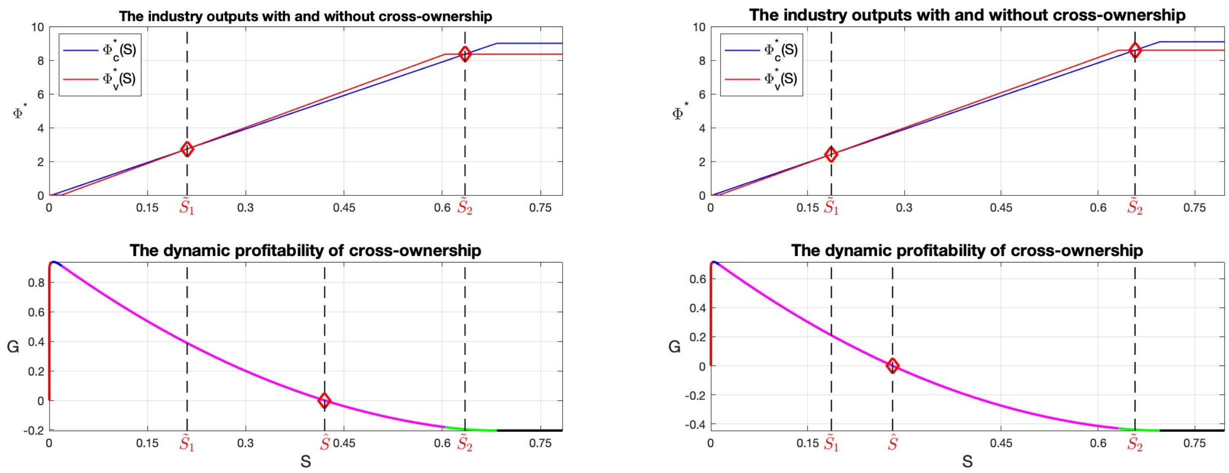
$$\frac{\partial V_o'(S)}{\partial v}\begin{cases} > 0 & \text{for } S_1 < S < S_{oR} \\ = 0 & \text{for } S = S_{oR} \\ < 0 & \text{for } S_{oR} < S < S_2 \end{cases}. \tag{12}$$

[11] Indeed, for any $n \geq 2k-1$, $G < 0$ for all $v \in (0, \frac{1}{k-1})$ in the static case; see Proposition 2 in Dai, Benchekroun and Long (2022).

Conditions (11) and (12) indicate that at a relatively large stock, i.e., $S \in (S_{iR}, S_2)$, engaging in cross-shareholdings will result in a decrease in both the insiders and outsiders' marginal valuation of the resource stock. At the same time, the reverse is true at a relatively small level of stock for $S \in (S_1, S_{oR})$. In addition, for some intermediate levels of stock, i.e., $S \in (S_{oR}, S_{iR})$, cross-ownership between rival firms leads to an increase in the cross-owners' marginal valuation of the resource stock but a reduction in the resource rent of the outsiders.

When a subset of competitors engage in rival cross-shareholdings, this results in an increase of the resource rents for these participants but also for those outsiders if the initial resource stock is relatively small, i.e., $S \in (S_1, S_{oR})$. Unlike in the static case where outsiders respond aggressively by increasing their production to mitigate any profit gains of insiders through output reduction due to ownership links, the existence of resource rent could attenuate such an increase and might even lead to a reduction in outsiders' production. This latter case can well occur when $S \in (S_1, \tilde{S})$, as discussed earlier in Proposition 3. Consequently, this engenders a more cautious response from outsiders, in which the optimal production of an outsider firm $o \in O$, given its rivals' production, tends to be lower compared to the static scenario where resource rents are absent. The 'moderation' of the outsiders' response to a reduction in production by the insiders, influenced by the existence of the resource rent, elucidates the fact that there exists a stock range within which a symmetric cross-ownership can be profitable, despite such rival cross-shareholdings being unprofitable in the corresponding static equilibrium framework. A similar result can be found in Benchekroun and Gaudet (2015), who find that there always exists an interval of the asset's stock such that any merger is profitable. It should be noted that the above analysis is conducted for the least possible profitable case: $k = 2$ and $n = 3$. Similar findings can be also obtained using other combinations $k$ and $n$ satisfying $\frac{k}{n} \leq \frac{k}{2k-1}$ that is strictly unprofitable for any admissible $v \in (0, \frac{1}{k-1})$, or any $\frac{k}{n} > \frac{k}{2k-1}$ that is unprofitable for $v \in (\bar{v}, \frac{1}{k-1})$ in the static framework.

Result 1 and Result 2 also help us clarify the question we left at the end of the last subsection, i.e., whether the output expansion resulting from cross-ownership can actually occur! For the case of $n = k$ in Corollary 2, the answer is obvious, since a symmetric industry-wide cross-ownership is always profitable for all $S$. Consequently, firms will always find it profitable to engage in cross-shareholdings for any $S$ in the first place, and thus there will always exist a range of $(\hat{S}_1, \hat{S}_2)$ such that the industry production increases as a result of profitable cross-ownership for any $n = k \geq 3$ and $v \in (0, \hat{v})$. As for the case of $n > k$ in Proposition 4, it is less straightforward. To illustrate this, we plot in Figure 11 both the industry outputs with and without cross-ownership and the dynamic profitability as a function of $S$, using the same parameter values as in Figure 8 but fixing $v = 0.15$ for $k = 6, n = 9$ in Figure 11(a) and for $k = 6, n = 10$ in Figure 11(b), respectively. It can be easily observed from Figure 11 that the range of resource stocks $(\tilde{S}_1, \tilde{S}_2)$ for which the industry output increases




(a) $k = 6, n = 9$ (b) $k = 6, n = 10$

Figure 11: The industry outputs and dynamic profitability as a function of $S$

following rival cross-shareholdings intersect with the interval of resource stocks $(0, \hat{S})$ for which a $k$-symmetric cross-ownership is profitable, where

$$G(k, n, v, \hat{S}) = (1 - (k-1)v)W(\hat{S}) - W_c(\hat{S}) = 0.$$

That is, for any $S \in (\tilde{S}_1, \hat{S})$, not only is it profitable for firms to engage in rival cross-shareholdings, but also the industry production will increase as a result of cross-ownership for any $n > k$ and $v \in (0, \frac{1}{k-1})$.

## 3.3 The short-run welfare implications

In the preceding analysis, we have illustrated the private incentives that motivate rival firms to engage in cross-shareholding. Additionally, we have demonstrated that industry output can increase following profitable rival cross-shareholdings. One direct implication of these results is that consumer surplus might also increase as a result of cross-ownership. This aspect holds significant relevance for discussions surrounding competition policies, as there is a growing call for more stringent regulations of these non-controlling minority shareholdings that are currently subject to a very lenient approach by antitrust authorities. In this subsection, we examine the welfare implications of cross-ownership in the context of a renewable resource industry, where welfare is defined as the sum of consumer surplus (CS) and producer surplus (PS) or industry profits. The latter is defined as the sum of the operating profits of the cross-ownership participants that belong to the subset $I$ of insiders and of the non-participants that belong to the subset $O$ of outsiders.

From our earlier analysis, we know that starting from any initial resource stock $S \in (\hat{S}_1, \hat{S}_2)$ when $n = k \geq 3$ and $v \in (0, \hat{v})$, or any $S \in (\tilde{S}_1, \hat{S})$ when $n > k$, the short-run industry output expands and thus consumer surplus increases following a profitable cross-ownership. We now show that industry profits can also go up for these cases. The producer surplus generated by the exploitation of the common property renewable resource under the $k$-symmetric cross-ownership structure is given by

$$PS_v = kV(S) + (n-k)V_o(S) = \begin{cases} \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{for } 0 \le S \le S_1 \\ \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S) & \text{for } S_1 < S \le S_2 \\ \frac{k\pi_i^v+(n-k)\pi_o^v}{r} & \text{for } S > S_2 \end{cases},$$

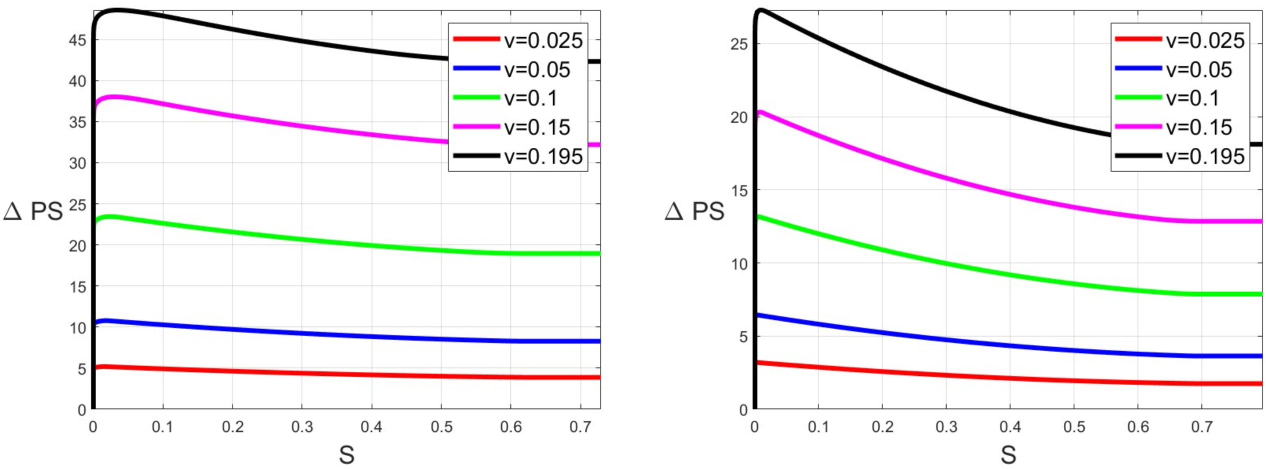
while the one without cross-ownership is denoted by

$$PS_c = nV_c = \begin{cases} n\left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \le S \le S_{1,N} \\ nW_c(S) & \text{for } S_{1,N} < S \le S_{2,N} \\ \frac{n\pi_c}{r} & \text{for } S > S_{2,N} \end{cases}.$$

Thus, the change in PS can be defined as

$$\Delta PS = PS_v - PS_c$$
$$= \begin{cases} \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - n\left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \le S < S_{1,N} \\ \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - nW_c(S) & \text{for } S_{1,N} \le S < S_1 \\ \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S) - nW_c(S) & \text{for } S_1 \le S < S_2 \\ \frac{k\pi_i^v+(n-k)\pi_o^v}{r} - nW_c(S) & \text{for } S_2 \le S < S_{2,N} \\ \frac{k\pi_i^v+(n-k)\pi_o^v}{r} - \frac{n\pi_c}{r} & \text{for } S \ge S_{2,N} \end{cases}.$$

Using the same parameter values as in Figure 8, we plot the PS change as a function of the initial stock $S$ for different levels of $v$ when $k=6, n=6$ in Figure 12(a) and when $k=6, n=10$ in Figure 12(b). It can be easily observed that for all $S$, the change in PS is




(a) $k=6, n=6$

(b) $k=6, n=10$

Figure 12: The change in PS as a function of $S$

positive. Simulations using many other combinations of $k, n$ and $v$ also show the same result. This indicates that for the above-mentioned two scenarios, a profitable cross-ownership can not only increase industry production and CS, but also boost industry profits, leading to a higher overall welfare.

**Result 3.** *Profitable rival cross-shareholdings can lead to a higher consumer surplus, producer surplus and welfare in the short run if any of the following scenarios occurs:*

*(i)* $n = k \geq 3$, $v \in (0, \hat{v})$ *and* $S \in (\hat{S}_1, \hat{S}_2)$, *where* $\hat{S}_1 \in (S_1, S_2)$,$\hat{S}_2 \in (S_2, S_{2,N})$ *and* $\hat{v} = \frac{(n+1)\left(r-2\delta+n(\delta-r)\right)}{(\delta-r)\left(n^2(n-2)\right)+2\delta-r+n\delta}$;

*(ii)* $n > k \geq 2$, *and* $S \in (\tilde{S}_1, \hat{S})$, *where* $\tilde{S}_1 \in (S_1, S_2)$ *and* $G(k, n, v, \hat{S}) = 0$ *with* $\hat{S} \in (\tilde{S}_1, S_2)$.

This result is in sharp contrast with the static oligopoly and cross-ownership theory, according to which cross-ownership always leads to a welfare loss in the absence of any efficiency gains. Indeed, when firms engage in rival cross-shareholdings, they tend to compete less aggressively with each other and thus unilaterally reduce their production. This happens because any increase in the acquiring firm's activities could diminish the returns from its shareholdings in the target firm. Although the outsider firms that are not involved in cross-shareholdings respond by increasing their production, the reduction in outputs from the cross-owners more than offsets this increase. As a result, the total industry output falls and the market price increases. While this benefits the industry by boosting profits, it reduces consumer surplus. However, the loss in consumer surplus dominates the gains in industry profits, resulting in an overall welfare loss. But in our dynamic framework where an oligopoly exploits a common pool productive asset, the presence of the resource rent effect dominates this standard static market power effect conferred by cross-ownership for the above-stated scenarios in Result 3. The former increases production, which outweighs the output reduction induced by the latter, leading to a higher industry output and thus CS. The industry profits also increase, because both insiders and outsiders expand their production at a slightly decreased price due to a relatively moderate response. Consequently, the social welfare is higher in the short run following the profitable rival cross-shareholding activities. This result thus suggests that competition authorities should be cautious when ruling in the renewable resource sector, as cross-ownership may turn out to be welfare-improving.

# 4 The long-run impact of cross-ownership

In this section, we explore the effects of cross-ownership on the long-run steady-state resource stocks, industry outputs, profitability, and social welfare. More specifically, we compare the outcomes under the $k$- symmetric cross-ownership structure ($v > 0$)

and the one without cross-ownership ($v = 0$), and then we characterize conditions under which cross-ownership may lead to an increase in the industry output and social welfare at the steady state.

## 4.1 The effects on stationary resource stocks and industry outputs

We start with the analysis on the impact of cross-ownership on the productive asset's stock at the steady state and the industry's production. First, note that

**Lemma 4.** *For any $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$, $S_1^\infty > S_{1,N}^\infty$, $S_2^\infty < S_{2,N}^\infty$ and $S_3^\infty < S_{3,N}^\infty$.*

*Proof.* See Appendix H. □

As discussed earlier in Corollary 1, the steady state level of the asset depends crucially on the initial resource stock. We can thus distinguish the following three cases. First, let us consider

$$\delta S_y < Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} < Q_c = \frac{an}{b(n+1)} \tag{LC1}$$

in which there is only one positive stationary asset stock in both the cases with and without cross-ownership given by $S_1^\infty$ and $S_{1,N}^\infty$, respectively, as shown in Figure 13. From Lemma 4, we know that $S_1^\infty > S_{1,N}^\infty$ for all $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$. There-

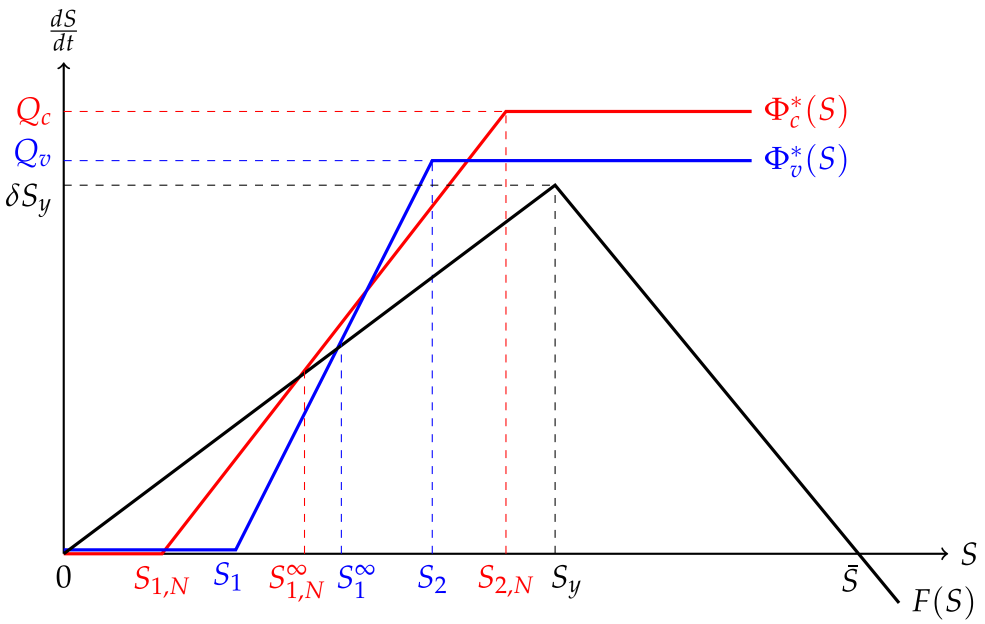



Figure 13: The stationary stock when $\delta S_y < Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} < Q_c = \frac{an}{b(n+1)}$

fore, the long-run industry outputs with and without cross-ownership are respectively given by

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^\infty, \quad \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty, \quad \forall\ S_0 > 0$$

with

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)).$$

That is, following the rival cross-shareholding activities, both the productive asset's stock at the steady state and the industry's production has gone up.

Next, we consider the situation

$$Q_c = \frac{an}{b(n+1)} > \delta S_y > Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} \tag{LC2}$$

in which there are three positive stationary stocks in the case of cross-ownership given by $S_1^\infty$, $S_2^\infty$ and $S_3^\infty$ respectively, while there is only one positive stationary stock in the case of no cross-ownership given by $S_{1,N}^\infty$, as shown in Figure 14. Clearly, we have

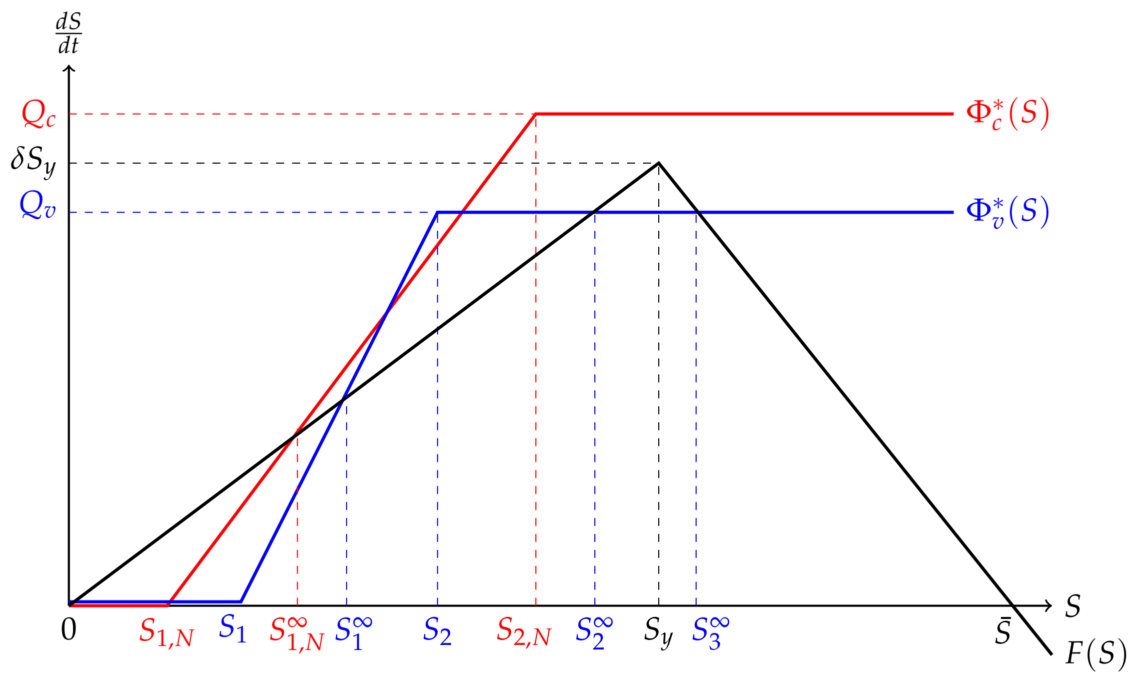



Figure 14: The stationary stocks when $Q_c = \frac{an}{b(n+1)} > \delta S_y > Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$

$$S_3^\infty > S_2^\infty > S_1^\infty > S_{1,N}^\infty, \quad \forall\, 2 \le k \le n,\ 0 < v < \frac{1}{k-1}.$$

That is, regardless of the initial resource stock, the asset's stock will converge to a larger steady-state level following the cross-ownership activities. For any $S_0 \in (0, S_2^\infty)$, we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^\infty > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty,$$

and for any $S_0 > S_2^\infty$, we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^\infty}{1-S_y}\right) = Q_v > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty.$$

Therefore, we can also claim that cross-ownership will result in a higher steady-state resource stock and industry output in this scenario.

Finally, let us consider

$$\delta S_y > Q_c = \frac{an}{b(n+1)} > Q_v = \frac{((k+n-k^2)v+n)a}{((k+n+1-k^2)v+n+1)b} \tag{LC3}$$

in which there are three positive steady-state stocks in both the case with cross-ownership and without cross-ownership as shown in Figure 15. The stationary resource stocks

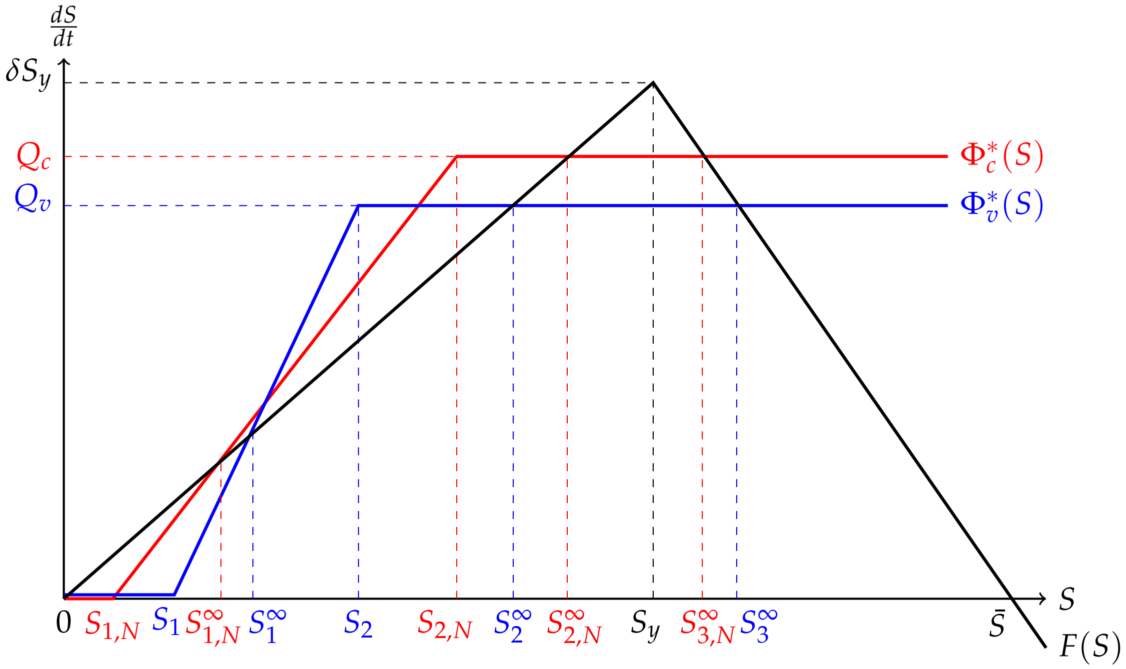



Figure 15: The stationary stocks when $\delta S_y > Q_c = \frac{an}{b(n+1)} > Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$

with cross-ownership are given by $S_1^\infty$, $S_2^\infty$ and $S_3^\infty$, while those without cross-ownership are denoted by $S_{1,N}^\infty$, $S_{2,N}^\infty$ and $S_{3,N}^\infty$, respectively. From Lemma 4 and Figure 15, we know that

$$S_{1,N}^\infty < S_1^\infty < S_2^\infty < S_{2,N}^\infty < S_{3,N}^\infty < S_3^\infty, \quad \forall\, 2 \le k \le n,\ 0 < v < \frac{1}{k-1}.$$

For any $S_0 \in (0, S_2^\infty)$,

$$\lim_{t\to\infty} S_v^*(t) = S_1^\infty > \lim_{t\to\infty} S_c^*(t) = S_{1,N}^\infty,$$

and thus

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^\infty > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty.$$

In addition, for any $S_0 \in (S_2^\infty, S_{2,N}^\infty)$,

$$\lim_{t\to\infty} S_v^*(t) = S_3^\infty > \lim_{t\to\infty} S_c^*(t) = S_{1,N}^\infty,$$

and thus we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^\infty}{1-S_y}\right) = Q_v > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty.$$

Furthermore, for any $S_0 > S_{2,N}^\infty$,

$$\lim_{t\to\infty} S_v^*(t) = S_3^\infty > \lim_{t\to\infty} S_c^*(t) = S_{3,N}^\infty,$$

and thus

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^\infty}{1-S_y}\right) = Q_v < \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_y \left(\frac{1-S_{3,N}^\infty}{1-S_y}\right) = Q_c.$$

To conclude in this scenario, regardless of the initial resource stock, the stationary asset stock is always higher following cross-ownership, and there exist cases where the long-run industry output increases as a result of cross-ownership.

Based on the findings from all these three possible cases, we can therefore summarize in the following propositions the impact of cross-ownership on the long-run resource stocks and the industry's production.

**Proposition 5.** *Regardless of the initial resource stock, cross-ownership results in a larger steady-state level of the productive asset's stock.*

**Proposition 6.** *For any $2 \le k \le n$ and $0 < v < \frac{1}{k-1}$,*

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) > \lim_{t\to\infty} \Phi_c^*(S_c^*(t))$$

*if one of the following conditions holds:*

$$(i)\ \delta S_y < Q_c = \frac{an}{b(n+1)},$$

*or*

$$(ii)\ \delta S_y > Q_c = \frac{an}{b(n+1)}, \quad \textit{and} \quad S_0 \in (0, S_{2,N}^\infty).$$

Proposition 6 demonstrates that at the stationary equilibrium, there exist conditions under which cross-shareholdings between rival firms can lead to an increase in the industry output. The result is quite surprising and sharply contrasts with the predictions of static oligopoly and cross-ownership theory. Indeed, when a subset of firms partially internalize their previous rivalry due to their ownership links, they unilaterally reduce their production. But in terms of strategic substitutes in Cournot competition, other non-participating firms will respond by expanding their production, aiming to capture a larger market share. Nonetheless, the reduction in output by cross-owners outweighs the production expansion by outsiders, resulting in an overall decrease in industry production. In our context where an oligopoly exploits a productive asset, this static result would hold if the initial resource stock is large enough and the implicit growth rate exceeds a certain threshold (i.e., $S_0 > S_{2,N}^{\infty}$ and $\delta > \frac{an}{bS_y(n+1)}$). However, when the implicit growth rate falls below a certain threshold (i.e., $\delta < \frac{an}{bS_y(n+1)}$) or the initial resource stock is small enough (i.e.,$S_0 < S_{2,N}^{\infty}$), rival cross-shareholdings can lead to an increase in the industry's production. This is because in our dynamic framework with a productive common asset, cross-ownership between rival firms influences the industry's exploitation rate through two channels: the output market and the interaction at the resource level. The former is the traditional channel through which reduced competition in the output market due to ownership links makes the industry output fall, while the latter is specific to the renewable resource industry whereas cross-ownership between rival firms results in a larger long-run stock of the asset and consequently allows for greater extraction by the industry. Proposition 6 shows that the latter impact of cross-ownership dominates the former one.

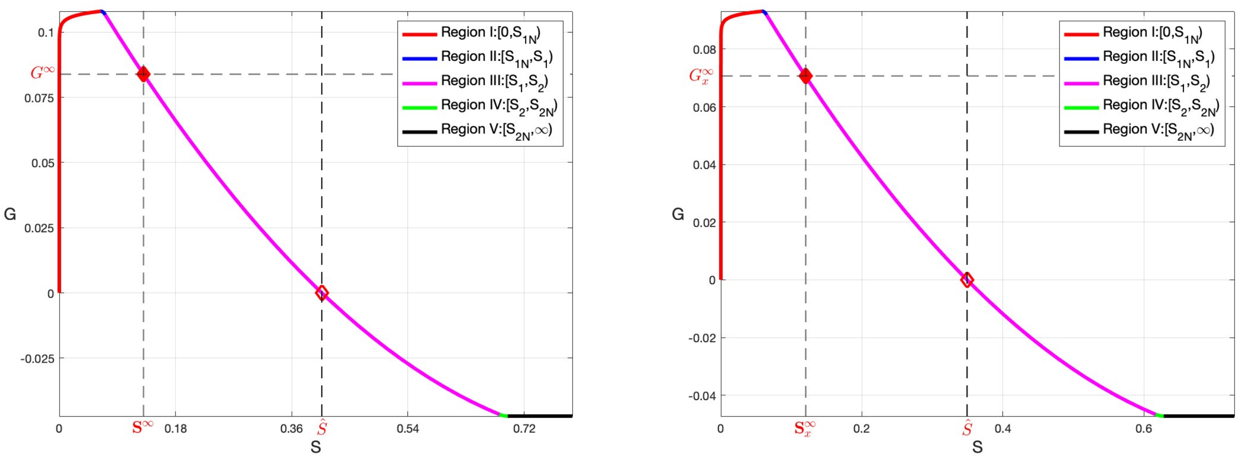
## 4.2 The dynamics of cross-ownership and long-run profitability

Despite our solid explanations on why cross-ownership could lead to an increased industry output in the long run, one might still question whether such a scenario would ever materialize, as firms may not find it profitable to engage in cross-shareholdings in the transition towards the steady-state level of the stock. To illustrate this, it is sufficient to give some examples. For simplicity, we limit our analysis to the first situation $\delta S_y < Q_c = \frac{an}{b(n+1)}$ where there is only one steady state before cross-ownership.

First, consider $\delta S_y < Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} < Q_c = \frac{an}{b(n+1)}$ in which there is only one steady state before and after cross-ownership as shown in Figure 13. The stationary asset stocks are given by $S_{1,N}^{\infty}$ and $S_1^{\infty}$, both falling into region III: $S \in [S_1, S_2)$ with $S_1 < S_{1,N}^{\infty} < S_1^{\infty} < S_2$, and the associated long-run industry outputs are $\delta S_{1,N}^{\infty}$ and $\delta S_1^{\infty}$, respectively. Therefore, the profitability function is given by

$$G^{\infty}(k,n,v,\boldsymbol{S}^{\infty}) = (1-(k-1)v)W(S_1^{\infty}) - W_c(S_{1,N}^{\infty}),$$

where $\boldsymbol{S}^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$. To show that $G^{\infty}(k,n,v,\boldsymbol{S}^{\infty})$ can be positive, we refer back to the least possibly profitable case of $k=2, n=3$. Using parameter values of $a=5, b=0.5, r=0.15, S_y=0.75$, we consider $v=0.1$ that is strictly unprofitable in the static framework and choose $\delta=9$ that satisfies both Assumption 1 and condition (LC1). Figure 16(a) reproduces the dynamic profitability as a function of $S$ in the short run and adds the long-run profitability when $\boldsymbol{S}^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\} = \{0.2634, 0.2782\}$ and $G^{\infty} = 0.0839$. In the short run, for any $S_0 < \hat{S} = 0.4065$, the profitability of cross-ownership is positive, while for any $S_0 > \hat{S}$, each of the two firms will find it unprofitable to hold a 10% share of the other in a triopoly industry. That is, a symmetric cross-ownership between two firms that is profitable at $t=0$ if $S_0 < \hat{S}$ will remain profitable throughout the transition to the steady-state level of the stock. Moreover, the unprofitable cross-ownership at $t = 0$ for $S > \hat{S}$ can become profitable as the stock evolves to the steady-state value.




(a) $Q_c > Q_v > \delta S_y$

(b) $Q_c > \delta S_y > Q_v$

Figure 16: Transition from short-run to long-run profitability

Now, let us look at the case of $Q_c = \frac{an}{b(n+1)} > \delta S_y > Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$ in which there is only one positive stationary stock before cross-ownership given by $S_{1,N}^\infty$ and there are three positive stationary stocks after cross-ownership given by $S_1^\infty$, $S_2^\infty$ and $S_3^\infty$ respectively, as shown in Figure 14. We continue to use the example of $k = 2, n = 3$ and $v = 0.1$ but set $\delta = 9.95$ that satisfies both Assumption 1 and condition (LC2), while keeping the other parameter values unchanged. In this case, the pair of long-run steady-state stocks can be either $\boldsymbol{S}_x^\infty = \{S_{1,N}^\infty, S_1^\infty\} = \{0.2395, 0.2528\}$ if $S_0 < S_2^\infty = 0.7418$ or $\boldsymbol{S}_y^\infty = \{S_{1,N}^\infty, S_3^\infty\} = \{0.2395, 0.7527\}$ when $S_0 > S_2^\infty = 0.7418$ and the corresponding profitability is thus either

$$G_x^\infty(k, n, v, \boldsymbol{S}_x^\infty) = (1 - (k-1)v)W(S_1^\infty) - W_c(S_{1,N}^\infty) = 0.0706 > 0,$$

or

$$G_y^\infty(k, n, v, \boldsymbol{S}_y^\infty) = \frac{\pi_i^v}{r} - W_c(S_{1,N}^\infty) = 0.6155 > 0.$$

A similar graph is produced in Figure 16(b) for the short-run profitability as a function of $S$ but only adds the long-run profitability $G_x^\infty$ at $\boldsymbol{S}_x^\infty = \{S_{1,N}^\infty, S_1^\infty\}$. So at $t = 0$, a symmetric cross-ownership between 2 firms in a 3-firm industry will profit from this mutual shareholding of 10% if $S_0 < \hat{S} = 0.3492$, and they will continue to find it profitable in the transition to the new steady state at $\boldsymbol{S}_x^\infty = \{S_{1,N}^\infty, S_1^\infty\}$. However, if the initial stock is $S_0 \in (\hat{S}, S_2^\infty)$ or $S_0 > S_2^\infty$, the profitability of cross-ownership is strictly negative in the short run, but in the transition towards the new steady state, this original unprofitability will transform into positive profitability with the former converging to $\boldsymbol{S}_x^\infty$ and the latter to $\boldsymbol{S}_y^\infty$.

Simulations using many other combinations of $k, n, v$ can also find such results.

These results thus confirm that firms will find it profitable to engage in cross-shareholdings in the transition to the steady state of the stocks. Consequently, the long-run expansion of industry production becomes a viable prospect, suggesting the potential for an increase in consumer surplus in the long run as a result of cross-ownership.

## 4.3 The long-run welfare implications

Now let us turn to the comparison of CS, PS and welfare at the stationary equilibrium. By Proposition 6, we know that the stationary CS is higher when firms engage in rival cross-shareholdings than in the case without cross-ownership for $\delta S_y < Q_c = \frac{an}{b(n+1)}$, or $\delta S_y > Q_c = \frac{an}{b(n+1)}$ and $S_0 \in (0, S_{2,N}^{\infty})$. Thus, it remains to show that stationary industry profits can also be higher in these scenarios as a result of cross-ownership. We discuss the respective three cases in the following.

1. If $\delta S_y < Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} < Q_c = \frac{an}{b(n+1)}$ in which there is only one stable steady state before and after cross-ownership, then the pair of stationary asset stocks is given by $\boldsymbol{S}^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$. Thus, the change in PS at the stationary equilibrium is given by

$$\Delta PS(\boldsymbol{S}^{\infty}) = \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S_1^{\infty}) - nW_c(S_{1,N}^{\infty}).$$

2. If $Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} < \delta S_y < Q_c = \frac{an}{b(n+1)}$ in which there is only one positive stable stationary stock before cross-ownership and there are two positive stable stationary stocks after cross-ownership, then the pair of steady-state stocks can be either $\boldsymbol{S}_x^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$ or $\boldsymbol{S}_y^{\infty} = \{S_{1,N}^{\infty}, S_3^{\infty}\}$. The stationary PS change in the former is

$$\Delta PS(\boldsymbol{S}_x^{\infty}) = \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S_1^{\infty}) - nW_c(S_{1,N}^{\infty}),$$

while that in the latter is given by

$$\Delta PS(\boldsymbol{S}_y^{\infty}) = PS_v(S_3^{\infty}) - PS_c(S_{1,N}^{\infty}) = \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - nW_c(S_{1,N}^{\infty})$$

3. Finally, if $\delta S_y > Q_c = \frac{an}{b(n+1)}$ and $S_0 \in (0, S_{2,N}^{\infty})$, the pair of stationary stocks is either $\boldsymbol{S}_z^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$ or $\boldsymbol{S}_w^{\infty} = \{S_{1,N}^{\infty}, S_3^{\infty}\}$, with the PS change at the stationary equilibrium given by

$$\Delta PS(\boldsymbol{S}_z^{\infty}) = \frac{(1-(k-1)v)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S_1^{\infty}) - nW_c(S_{1,N}^{\infty}),$$

and

$$\Delta PS(S_w^\infty) = PS_v(S_3^\infty) - PS_c(S_{1,N}^\infty) = \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - nW_c(S_{1,N}^\infty),$$

respectively. It should be noted that $S^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ is not the same as $S_x^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ or $S_z^\infty = \{S_{1,N}^\infty, S_1^\infty\}$, as these cases correspond to different initial conditions when $\delta S_y < Q_v < Q_c$, $Q_v < \delta S_y < Q_c$ and $Q_c < \delta S_y$. The same applies to $S_y^\infty$ and $S_w^\infty$.

We now show that the PS change at the stationary equilibrium is always positive, irrespective of the initial conditions, and thus we can establish the following result:

**Proposition 7.** *Profitable rival cross-shareholdings can lead to a higher consumer surplus, producer surplus and welfare at the stationary equilibrium for* $\delta S_y < Q_c = \frac{an}{b(n+1)}$, *or* $\delta S_y > Q_c = \frac{an}{b(n+1)}$ *and* $S_0 \in (0, S_{2,N}^\infty)$.

*Proof.* See Appendix I. □

Proposition 7 indicates that cross-ownership can turn out to be welfare-improving in the long run. This result, combined with our findings in Result 3, suggests that welfare can increase as a result of cross-shareholdings both in the short run and long run. Therefore, antitrust authorities should exercise caution when regulating renewable resource industries, as strict policies that restrict cooperation among users of common-pool renewable resources could ultimately harm consumers and society. Unintentionally, these measures might produce the exact opposite effect of what is intended.

# 5 Conclusion

In this paper, we have proposed a dynamic game of exploitation of a productive asset by agents who subsequently sell the outcomes of their endeavours in an oligopolistic market where a subset of the oligopolists owns a share in each other's profits. A Markov Perfect Nash Equilibrium of the game is constructed and used to analyze the impact of cross-ownership on the equilibrium production strategies, the steady state resource stocks, the profitability of cross-ownership, and social welfare. We show that there exists an interval of the renewable resource stock for which a symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are unprofitable in the corresponding static equilibrium framework. Moreover, we demonstrate that cross-ownership may not only lead to a higher market output and social welfare in the short run, but also a higher steady-state stock, industry production, and greater social welfare in the long run. These findings thus highlight the unique feature of the renewable resource industries and suggest that antitrust authorities should perform a specific examination when dealing with industries with stock dynamics.

We have focused on cross-ownership arrangements that are fixed at the beginning of the game, which made it easier to compare them with the corresponding static framework. However, if the decision to participate in cross-shareholdings were endogenous, we might expect more firms to join these "partial mergers". Yet, the endogenization of cross-ownership formation in this context remains complex and challenging, and we leave it for future research for which our model could serve as a useful starting point. Another possible extension could involve integrating pollution externalities into our analysis, as cross-shareholdings might affect both resource stocks and environmental quality. This aspect is particularly relevant because the drive for economic gain can lead to environmental degradation, presenting significant sustainability challenges. Understanding firms' strategic behaviour in this context is thus crucial for effective resource management and guiding regulatory policies (Vardar and Zaccour, 2020; Feichtinger et al., 2022).

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# Appendices

## A Proof of C1–C4

*Proof.* For the first inequality, it directly follows that

$$\delta > \frac{r\left[\left(k(k-1)v - n(1+v)\right)^2 + (1+v)^2\right]}{2(1+v)^2} = \frac{r\Big(k(k-1)v - n(1+v)\Big)^2}{2(1+v)^2} + \frac{r}{2} > \frac{r}{2}.$$

For the second inequality, from $\delta > \frac{r\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{2(1+v)^2}$, we have

$$\delta - r > \frac{r\left[\Big(k(k-1)v - n(1+v)\Big)^2 - (1+v)^2\right]}{2(1+v)^2}.$$

We now show that

$$\Big(k(k-1)v-n(1+v)\Big)^2-(1+v)^2 = \Big(k(k-1)v-(n-1)(1+v)\Big)\Big(k(k-1)v-(n+1)(1+v)\Big) > 0.$$

Since

$$k \le n \Rightarrow k(k-1)v \le n(n-1)v,$$

we know that

$$\Big(k(k-1)v-(n-1)(1+v)\Big) \le n(n-1)v-(n-1)(1+v) = (n-1)\big((n-1)v-1\big) < 0,$$

$$\Big(k(k-1)v-(n+1)(1+v)\Big) \le n(n-1)v-(n+1)(1+v) = n\Big((n-2)v-1\Big)-(1+v) < 0$$

and thus

$$\delta - r > 0.$$

For the third inequality, note that from $\delta > \frac{r\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{2(1+v)^2}$,

$$(2\delta - r)(1+v)^2 > r\big(k(k-1)v - n(1+v)\big)^2,$$

$$\iff (2\delta - r)(1+v) > r\frac{\left(k(k-1)v - n(1+v)\right)^2}{1+v}.$$

We now show that

$$r\frac{\left(k(k-1)v - n(1+v)\right)^2}{1+v} > -r\big(k(k-1)v - n(1+v)\big)$$

$$\iff \Big(k(k-1)v - n(1+v)\Big)^2 > n(1+v)^2 - kv(k-1)(1+v)$$

$$\iff k^2(k-1)^2v^2 - 2knv(k-1)(1+v) + n^2(1+v)^2 - n(1+v)^2 + kv(k-1)(1+v) > 0$$

$$\iff k^2(k-1)^2v^2 - (2n-1)kv(k-1)(1+v) + n(n-1)(1+v)^2 > 0$$

$$\iff \Big(k(k-1)v - n(1+v)\Big)\Big(k(k-1)v - (n-1)(1+v)\Big) > 0,$$

which is true, since $k \leq n$, we have

$$k(k-1)v - n(1+v) \leq n(n-1)v - n(1+v) = n\Big((n-2)v - 1\Big) < 0,$$

$$k(k-1)v - (n-1)(1+v) \leq n(n-1)v - (n-1)(1+v) = (n-1)\Big((n-1)v - 1\Big) < 0.$$

Therefore,

$$(2\delta - r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big) > 0.$$

Finally, for the last condition, note that both $\mathcal{H}_1(k,n,v) \equiv \frac{r\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{2(1+v)^2}$ and $\mathcal{H}_2(k,n,v) \equiv \frac{a\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{bS_y\left((k+n+1-k^2)v+n+1\right)^2}$ are strictly decreasing functions in $v$:

$$\frac{\partial \mathcal{H}_1(k,n,v)}{\partial v} = \frac{k(k-1)r\Big(k(k-1)v - n(1+v)\Big)}{(1+v)^3} < 0,$$

$$\frac{\partial \mathcal{H}_2(k,n,v)}{\partial v} = \frac{2k(k-1)a\Big(k(k-1)v - (n-1)(1+v)\Big)}{bS_y\Big((k+n+1-k^2)v+n+1\Big)^3} < 0.$$

Thus, for any $v \in (0, \frac{1}{k-1})$, Assumption 1 is equivalent to

$$\delta > \delta_0 \equiv max\left\{\mathcal{H}_1(k,n,0), \mathcal{H}_2(k,n,0)\right\} = max\left\{\frac{r(n^2+1)}{2}, \frac{a(n^2+1)}{bS_y(n+1)^2}\right\}.$$

□

# B Proof of Proposition 1

*Proof.* The vector $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ constitutes a MPNE if there exist $n$ continuously differentiable value functions $(V_i, \cdots, V_i, V_o, \cdots, V_o)$ such that the functions

$\phi_i^*(S)$ and $\phi_o^*(S)$ are solutions to the problems

$$rV_i(S) = max_{\phi_i}\left\{\frac{a - b\phi_{-i} - b\phi_i}{(1+v)\big(1-(k-1)v\big)}\left(\big(1-(k-2)v\big)\phi_i + v\sum_{m\in I\setminus i}\phi_m\right) + V_i'(S)(F(S) - \phi_{-i} - \phi_i)\right\}, \tag{13}$$

for $i \in I = \{1, 2, \cdots, k\}$ and

$$rV_o(S) = max_{\phi_o}\left\{(a - b\phi_{-o} - b\phi_o)\phi_o + V_o'(S)(F(S) - \phi_{-o} - \phi_o)\right\}, \tag{14}$$

for $o \in O = \{k+1, \cdots, n\}$. Consider the following value functions

$$V_i(S) = \begin{cases} \left(\frac{S}{S_i^1}\right)^{\frac{r}{\delta}} W(S_i^1) & \text{for } 0 \le S \le S_i^1, \\ W(S) & \text{for } S_i^1 < S \le S_i^2, \\ \frac{\Pi_i}{r} & \text{for } S > S_i^2, \end{cases}$$

$$V_o(S) = \begin{cases} \frac{(1+v)\left(1-(k-1)v\right)}{1-(k-2)v}\left(\frac{S}{S_o^1}\right)^{\frac{r}{\delta}} W(S_o^1) & \text{for } 0 \le S \le S_o^1, \\ \frac{(1+v)\left(1-(k-1)v\right)}{1-(k-2)v} W(S) & \text{for } S_o^1 < S \le S_o^2, \\ \frac{\Pi_o}{r} & \text{for } S > S_o^2, \end{cases}$$

where

$$\Pi_i = \frac{\pi_i^v}{1-(k-1)v} = \frac{1}{1-(k-1)v}\frac{(1+v)\big(1-(k-2)v\big)a^2}{\big((k+n+1-k^2)v+n+1\big)^2 b},$$

$$\Pi_o = \pi_o = \frac{(1+v)^2 a^2}{\big((k+n+1-k^2)v+n+1\big)^2 b} \equiv \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}\Pi_i,$$

$$W(S) = \frac{A}{2}S^2 + BS + C,$$

with

$$A = \frac{b(r-2\delta)\big(1-(k-2)v\big)\Big((k+n+1-k^2)v+n+1\Big)^2}{2(1+v)\big(1-(k-1)v\big)\Big((k+n-k^2)v+n\Big)^2},$$

$$B = \frac{a(2\delta-r)\big(1-(k-2)v\big)\left[\Big(k(k-1)v - n(1+v)\Big)^2 + (1+v)^2\right]}{2\delta(1+v)\big(1-(k-1)v\big)\Big((k+n-k^2)v+n\Big)^2},$$

$$C = \frac{a^2\big(1-(k-2)v\big)C_1C_2}{4br\delta^2(1+v)\big(1-(k-1)v\big)\Big((k+n-k^2)v+n\Big)^2\Big((k+n+1-k^2)v+n+1\Big)^2},$$

$$C_1 = \Big(2\delta + rk(k-1)\big(2n-k(k-1)\big) - r(n^2+1)\Big)v^2 + \Big(4\delta + 2rkn(k-1) - 2r(n^2+1)\Big)v + 2\delta - r(n^2+1),$$

$$C_2 = \Big((2\delta - r)\big(k(k-1) - n\big)^2 - r\Big)v^2 + \Big(4\delta n^2 + 2kn(k-1)(r-2\delta) - 2r(n^2+1)\Big)v + 2\delta n^2 - r(n^2+1),$$

$$S_i^1 = \frac{\Big(1-(k-2)v\Big)a - (1+v)\Big(1-(k-1)v\Big)B}{(1+v)\Big(1-(k-1)v\Big)A} = S_o^1, \quad S_i^2 = -\frac{B}{A} = S_o^2.$$

In the following, we show that (i) the value functions $V_i(S)$ and $V_o(S)$ are continuously differentiable; (ii) the functions $\phi_i^*(S)$ and $\phi_o^*(S)$ given by (5) and (6) are solutions to the problems (13) and (14).

First, note that there exists a unique relationship between $V_i(S)$ and $V_o(S)$ such that

$$V_o(S) = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v} V_i(S).$$

Therefore, we only need to prove that $V_i(S)$ is continuously differentiable. Clearly, the value function $V_i(S)$ is continuously differentiable over $(0, S_i^1), (S_i^1, S_i^2)$ and $(S_i^2, \infty)$, respectively, with

$$V_i'(S) = \begin{cases} \frac{r}{\delta S_i^1}\left(\frac{S}{S_i^1}\right)^{\frac{r}{\delta}-1} W(S_i^1) & \text{for } 0 \le S \le S_i^1, \\ W'(S) & \text{for } S_i^1 < S \le S_i^2, \\ 0 & \text{for } S > S_i^2, \end{cases}$$

We then need to check that the function $V_i(S)$ is continuously differentiable at $S_i^1$ and $S_i^2$. We have

$$\lim_{S \to S_i^1, S < S_i^1} V_i(S) = W(S_i^1) = \lim_{S \to S_i^1, S > S_i^1} V_i(S),$$

and

$$\lim_{S \to S_i^2, S < S_i^2} V_i(S) = W(S_i^2) = \Pi_i = \lim_{S \to S_i^2, S > S_i^2} V_i(S).$$

Thus, $V_i(S)$ is continuous at both $S_i^1$ and $S_i^2$. Also, note that

$$\lim_{S \to S_i^1, S < S_i^1} V_i'(S) = \frac{r}{\delta S_i^1} W(S_i^1).$$

It can be easily checked that

$$\frac{r}{\delta S_i^1} W(S_i^1) = W'(S_i^1) = \frac{1-(k-2)v}{(1+v)(1-(k-1)v)} a.$$

Thus, we must have

$$\lim_{S \to S_i^1, S < S_i^1} V_i'(S) = W'(S_i^1) = \lim_{S \to S_i^1, S > S_i^1} V_i'(S),$$

i.e., $V_i'(S)$ is continuous at $S_i^1$. Similarly, we have

$$\lim_{S\to S_i^2, S<S_i^2} V_i'(S) = 0 = \lim_{S\to S_i^2, S>S_i^2} V_i'(S).$$

So $V_i'(S)$ is continuous at both $S_i^1$ and $S_i^2$. Therefore, we can conclude that both the functions $V_i(S)$ and $V_o(S)$ are continuously differentiable over $[0,\infty)$.

Next, we show that $\phi_i^*(S)$ and $\phi_o^*(S)$ given by (5) and (6) are solutions to the problems (13) and (14), where $V_1(S) = V_2(S) = \cdots = V_k(S)$, and $V_{k+1}(S) = V_{k+2}(S) = \cdots = V_n(S)$. First, for $S \geq S_i^1$ and $S \geq S_o^1$, the system of problems (13) and (14) admit a pair of interior solutions. The HJB equation for firm $i \in I$ is

$$rV_i(S) = max_{q_i}\left\{\frac{a - b\sum_{j\neq i} q_j - bq_i}{(1+v)(1-(k-1)v)}\left((1-(k-2)v)q_i + v\sum_{m\in I\setminus i} q_m\right) + V_i'(S)(F(S) - \sum_{j\neq i} q_j - q_i)\right\},$$

and that for firm $o \in O$ is given by

$$rV_o(S) = max_{q_o}\left\{(a - b\sum_{j\neq o} q_j - bq_o)q_o + V_o'(S)(F(S) - \sum_{j\neq o} q_j - q_o)\right\}.$$

FOCs of the right-hand side yield

$$\frac{(a - b\sum_{j\neq i} q_j - bq_i)(1-(k-2)v) - b\left((1-(k-2)v)q_i + v\sum_{m\in I\setminus i} q_m\right)}{(1+v)(1-(k-1)v)} - V_i'(S) = 0,$$

and

$$a - b\sum_{j\neq o} q_j - 2bq_o - V_o'(S) = 0.$$

Symmetry yields

$$\frac{(a - bkq_i - b(n-k)q_o)(1-(k-2)v) - b(1+v)q_i}{(1+v)(1-(k-1)v)} - V_i'(S) = 0,$$

and

$$a - bkq_i - b(n-k+1)q_o - V_o'(S) = 0.$$

So the best response functions can be expressed as

$$q_i = \frac{(1-(k-2)v)a - (1-(k-2)v)(n-k)bq_o - (1+v)(1-(k-1)v)V_i'(S)}{\left[1 + k + \left(1 - k(k-2)\right)v\right]b},$$

$$q_o = \frac{a - bkq_i - V_o'(S)}{b(n-k+1)}.$$

Guess the value function as

$$V_i(S) = \frac{A}{2}S^2 + BS + C,$$

$$V_o(S) = \frac{D}{2}S^2 + ES + F,$$

then

$$V_i'(S) = AS + B,$$

$$V_o'(S) = DS + E.$$

So the FOCs become

$$(a - bkq_i - b(n-k)q_o)(1-(k-2)v) - b(1+v)q_i = (AS+B)(1+v)\big(1-(k-1)v\big),$$

and

$$a - bkq_i - b(n-k+1)q_o = DS + E.$$

Solving for $(q_i, q_o)$ yields

$$q_i^* = \frac{\Big(1-(k-2)v\Big)a - (1+v)\Big(1-(k-1)v\Big)(n-k+1)(AS+B) + \Big(1-(k-2)v\Big)(n-k)(DS+E)}{\Big((k+n+1-k^2)v+n+1\Big)b} \tag{15}$$

$$q_o^* = \frac{(1+v)a + (1+v)\Big(1-(k-1)v\Big)k(AS+B) - \Big[1+k+\Big(1-k(k-2)\Big)v\Big](DS+E)}{\Big((k+n+1-k^2)v+n+1\Big)b} \tag{16}$$

Substitute $q_i^*$ and $q_o^*$ into the HJB equations, then we have

$$r\left(\frac{A}{2}S^2 + BS + C\right) = \frac{a - bkq_i^* - b(n-k)q_o^*}{(1-(k-1)v)}q_i^* + (AS+B)(\delta S - kq_i^* - (n-k)q_o^*),$$

$$r\left(\frac{D}{2}S^2 + ES + F\right) = \Big(a - bkq_i^* - b(n-k)q_o^*\Big)q_o^* + (DS+E)(\delta S - kq_i^* - (n-k)q_o^*).$$

Using the undetermined coefficients technique (Dockner et al., 2000), we can solve for

$$A = \frac{b(r-2\delta)(1-(k-2)v)\Big((k+n+1-k^2)v+n+1\Big)^2}{2(1+v)(1-(k-1)v)\Big((k+n-k^2)v+n\Big)^2}, \quad D = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}A,$$

$$B = \frac{a(2\delta-r)(1-(k-2)v)\Big[\Big(k(k-1)v-n(1+v)\Big)^2 + (1+v)^2\Big]}{2\delta(1+v)(1-(k-1)v)\Big((k+n-k^2)v+n\Big)^2}, \quad E = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}B,$$

$$C = \frac{a^2(1-(k-2)v)C_1C_2}{4br\delta^2(1+v)(1-(k-1)v)\Big((k+n-k^2)v+n\Big)^2\Big((k+n+1-k^2)v+n+1\Big)^2},$$

$$C_1 = \Big(2\delta + rk(k-1)(2n-k(k-1)) - r(n^2+1)\Big)v^2 + \Big(4\delta + 2rkn(k-1) - 2r(n^2+1)\Big)v + 2\delta - r(n^2+1),$$

$$C_2 = \Big((2\delta - r)(k(k-1)-n)^2 - r\Big)v^2 + \Big(4\delta n^2 + 2kn(k-1)(r-2\delta) - 2r(n^2+1)\Big)v + 2\delta n^2 - r(n^2+1),$$

$$F = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}C.$$

Therefore, we must have

$$V_o'(S) = DS + E = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}(AS+B) = \frac{(1+v)(1-(k-1)v)}{1-(k-2)v}V_i'(S).$$

Substituting these coefficients back to (15) and (16) yields

$$\phi_i^*(S) = q_i^* = \frac{\Big(1-(k-2)v\Big)a - (1+v)\Big(1-(k-1)v\Big)(AS+B)}{\Big((k+n+1-k^2)v+n+1\Big)b},$$

$$\phi_o^*(S) = q_o^* = \frac{1+v}{1-(k-2)v}\frac{\Big(1-(k-2)v\Big)a - (1+v)\Big(1-(k-1)v\Big)(AS+B)}{\Big((k+n+1-k^2)v+n+1\Big)b}.$$

Moreover, it can be easily observed that there exists a unique relationship between $\phi_i^*$ and $\phi_o^*$ such that

$$\phi_o^*(S) = \frac{1+v}{1-(k-2)v}\phi_i^*(S).$$

The level of stocks $S_i^2$ and $S_o^2$ are determined such that $V_i(S)$ and $V_o(S)$ are continuously differentiable in the neighborhood of $S_i^2$ and $S_o^2$, respectively. Thus, we have

$$S_i^2 = -\frac{B}{A} = S_o^2.$$

Finally, for $S < S_i^1$ and $S < S_o^1$, the system of problems (13) and (14) admit a pair of corner solutions such that

$$\phi_i^*(S) = 0 = \phi_o^*(S).$$

Therefore, we must have

$$S_i^1 = \frac{(1-(k-2)v)a - (1+v)(1-(k-1)v)B}{(1+v)(1-(k-1)v)A} = S_o^1.$$

□

# C Proof of Corollary 1

*Proof.* The stationary asset stocks are characterized by

$$\frac{dS}{dt} = F(S) - k\phi_i^*(S) - (n-k)\phi_o^*(S) = F(S) - \Phi_v^*(S).$$

(i) If $F(S)_{max} = \delta S_y < \Phi_v^*(S)_{max} = kq_i^v + (n-k)q_o^v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$, there is one and only one positive root given by solving

$$\frac{dS}{dt} = \delta S - \left(k + (n-k)\frac{1+v}{1-(k-2)v}\right)\frac{(1-(k-2)v)a - \big(1-(k-2)v-(k-1)v^2\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b} = 0,$$

which yields

$$S_1^\infty = \frac{a\left[(2\delta - r)(1+v)^2 - r\Big(k(k-1)v - n(1+v)\Big)^2\right]}{b\delta\Big((k+n+1-k^2)v+n+1\Big)\left[(2\delta - r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big)\right]}.$$

To ensure $S_1^\infty > 0$, we would need

$$(2\delta - r)(1+v)^2 - r\Big(k(k-1)v - n(1+v)\Big)^2 > 0.$$

That is,

$$2\delta - r > \frac{r\Big(k(k-1)v - n(1+v)\Big)^2}{(1+v)^2},$$

or

$$\delta > \frac{r\left[\Big(k(k-1)v - n(1+v)\Big)^2 + (1+v)^2\right]}{2(1+v)^2},$$

which also guarantees that the condition (C3) is satisfied:

$$(2\delta - r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big) > 0.$$

Moreover, we have

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S > S_1^\infty,$$

and

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, S < S_1^\infty.$$

Therefore, for any initial asset's stock $S_0$, the asset's stock equilibrium path converges asymptotically to $S_1^\infty$, i.e., $S_1^\infty$ is globally stable.

(ii) If $F(S)_{max} = \delta S_y > \Phi_v^*(S)_{max} = kq_i^v + (n-k)q_o^v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$, there are three positive roots. One is given by $S_1^\infty$, and the other two are given by solving

$$\frac{dS}{dt} = \delta S - \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} = 0,$$

and
$$\frac{dS}{dt} = \delta S_y\left(\frac{1-S}{1-S_y}\right) - \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} = 0,$$

respectively. Solving for $S$ yields

$$S_2^\infty = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b\delta}, \quad S_3^\infty = 1 - \frac{\big((k+n-k^2)v+n\big)a(1-S_y)}{\big((k+n+1-k^2)v+n+1\big)b\delta S_y}.$$

Moreover, we have

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, 0 < S < S_1^\infty,$$

and
$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S_1^\infty < S < S_2^\infty.$$

Therefore, for any initial stock $S_0 \in (0, S_2^\infty)$, the asset's stock equilibrium path converges monotonically to $S_1^\infty$, i.e., $S_1^\infty$ is stable. Also, we have

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, S_2^\infty < S < S_3^\infty,$$

and
$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S > S_3^\infty.$$

Therefore, for any initial stock $S_0 \in (S_2^\infty, \infty)$, the asset's stock equilibrium path converges monotonically to $S_3^\infty$, i.e., $S_3^\infty$ is stable. The stationary asset stock $S_2^\infty$ is thus unstable.

□

# D Proof of Lemma 1 and Lemma 2

*Proof.* Note that

$$S_2(v) = \frac{a\left[\Big(k(k-1)v - n(1+v)\Big)^2 + (1+v)^2\right]}{b\delta\Big((k+n+1-k^2)v+n+1\Big)^2}, \; S_1(v) = \frac{a\left[(2\delta-r)(1+v)^2 - r\Big(k(k-1)v - n(1+v)\Big)^2\right]}{b\delta(2\delta-r)\Big((k+n+1-k^2)v+n+1\Big)^2},$$

and thus

$$\frac{\partial S_2(v)}{\partial v} = \frac{2ak(k-1)\Big(k(k-1)v-(n-1)(1+v)\Big)}{b\delta\Big((k+n+1-k^2)v+n+1\Big)^3},$$

$$\frac{\partial S_1(v)}{\partial v} = \frac{2ak(k-1)\Big[(2\delta-r)(1+v)-r\Big(k(k-1)v-n(1+v)\Big)\Big]}{b\delta(2\delta-r)\Big((k+n+1-k^2)v+n+1\Big)^3}.$$

Since $k \le n$, we have

$$k(k-1)v-(n-1)(1+v) \le n(n-1)v-(n-1)(1+v) = (n-1)\Big((n-1)v-1\Big) < 0,$$

and

$$k(k-1)v-n(1+v) \le n(n-1)v-n(1+v) = n\Big((n-2)v-1\Big) < 0,$$

$$\Rightarrow (2\delta-r)(1+v)-r\Big(k(k-1)v-n(1+v)\Big) > 0.$$

Therefore,

$$\frac{\partial S_1(v)}{\partial v} > 0, \quad \frac{\partial S_2(v)}{\partial v} < 0.$$

That is, for any $v > 0$, $S_1(v) > S_1(0) = S_{1,N}$ and $S_2(v) < S_2(0) = S_{2,N}$.

In addition, we have

$$\frac{\partial \Omega_o(v)}{\partial v} = \frac{k(k-1)(2\delta-r)\Big((n+2)(1+v)-kv(k-1)\Big)}{2\Big((k+n-k^2)v+n\Big)^3} > 0,$$

since $2\delta - r > 0$,

$$(n+2)(1+v)-kv(k-1) \ge (n+2)(1+v)-nv(n-1) = 2(1+v)+n\Big(1-(n-2)v\Big) > 0,$$

and

$$(k+n-k^2)v+n = n+nv-k(k-1)v \ge n-k+nv > 0.$$

Moreover,

$$\frac{\partial \Omega_i(v)}{\partial v} = \frac{(k-1)(r-2\delta)\Big[n(n+1)-k(n+2)+\Big(n(n+1)-k^2(n-k)-3k\Big)v\Big]}{2\Big((k+n-k^2)v+n\Big)^3}.$$

In the case of $n = k$,

$$\frac{\partial \Omega_i}{\partial v} = \frac{k(k-1)(2\delta - r)\Big(1-(k-2)v\Big)}{2\Big((2k-k^2)v+k\Big)^3} = \frac{(k-1)(2\delta - r)}{2k^2\Big(1-(k-2)v\Big)^2} > 0.$$

That is, for any $v > 0$ and $n = k$, $\Omega_i(v) > \Omega_i(0) = \Omega_c$. However, in the case of $n > k$, since $r - 2\delta < 0$,

$$n(n+1) - k(n+2) = n(n-k+1) - 2k \geq 2n - 2k > 0,$$

and

$$F(k,n) \equiv n(n+1) - k^2(n-k) - 3k > 0, \tag{C5}$$

we must have $\frac{\partial \Omega_i}{\partial v} < 0$. We now prove that condition (C5) is always true for all $n > k$. Note that $F(k,n)$ is a quadratic U-shaped function of $n$:

$$F(k,n) = n^2 + (1-k^2)n + k^3 - 3k,$$

with

$$\Delta(k) = (1-k^2)^2 - 4(k^3 - 3k) = k^4 - 2k^2 + 1 - 4k^3 + 12k.$$

Since

$$\Delta'(k) = 4k^3 - 12k^2 - 4k + 12 = 4(k+1)(k-1)(k-3)\begin{cases} < 0 & \text{if } k = 2 \\ = 0 & \text{if } k = 3 \\ > 0 & \text{if } k \geq 4 \end{cases},$$

we have the following cases:

1. If $k = 2$, $\Delta'(k) > 0$ and $\Delta(2) = 9 - 4 \times 2 = 1 > 0$;
2. If $k = 3$, $\Delta'(k) = 0$, and we have the minimum value: $\Delta(3) = 64 - 4 \times 18 = -8 < 0$;
3. If $k \geq 4$, $\Delta'(k) > 0$ and since $\Delta(4) = 225 - 4 \times 52 = 17 > 0$, we must have $\Delta(k) > 0$ for all $k > 4$.

That is,

1. when $k = 2$, $F(2,n) = n^2 - 3n + 2 = (n-1)(n-2)$ has two roots:

   $$n_1 = 1, \quad n_2 = 2.$$

   Therefore, for any $n > n_2 = k = 2$, we have $F(2,n) > 0$.

2. When $k = 3$, $F(k, n)$ has no real roots of $n$, and thus it is always positive, i.e.,

$$F(3, n) = n^2 - 8n + 18 = (n - 4)^2 + 2 > 0 \quad \forall\, n > k = 3.$$

3. However, when $k \geq 4$, $F(k, n)$ has two real roots $n_1$ and $n_2$:

$$n_2 = \frac{k^2 - 1 + \sqrt{\Delta(k)}}{2} > n_1 = \frac{k^2 - 1 - \sqrt{\Delta(k)}}{2}$$

   Thus, $F(k, n)$ is strictly positive for any $n > n_2$ or $n < n_1$, and negative for any $n \in (n_1, n_2)$. We now show that for any $n > k \geq 4$, $n > n_2$ always holds. Is it true that $n > k > n_2$? Yes, indeed!

$$k > n_2 = \frac{k^2 - 1 + \sqrt{\Delta(k)}}{2}$$

$$\iff (2k - k^2 + 1)^2 > (1 - k^2)^2 - 4(k^3 - 3k)$$

$$\iff 4k^2 + 4k(1 - k^2) + 4(k^3 - 3k) > 0$$

$$\iff 4k(k - 2) > 0$$

Therefore, for any $v > 0$ and $n > k$, we must have

$$\Omega_i(v) < \Omega_i(0) = \Omega_c = \Omega_o(0) < \Omega_o(v).$$

Finally, we show that

$$\frac{\partial q_i^v}{\partial v} = \frac{a(k-1)(k-n-1)}{b\Big((k+n+1-k^2)v + n + 1\Big)^2} < 0,$$

$$\frac{\partial q_o^v}{\partial v} = \frac{ak(k-1)}{b\Big((k+n+1-k^2)v + n + 1\Big)^2} > 0.$$

Thus, for any $v > 0$,

$$q_i^v(v) < q_i^v(0) = q_c = q_o^v(0) < q_o^v(v).$$

$\square$

## E Proof of Proposition 2

*Proof.* Note that when $n = k$,

$$S_2 = \frac{a\left[\left(k(k-1)v - n(1+v)\right)^2 + (1+v)^2\right]}{b\delta\left((k+n+1-k^2)v+n+1\right)^2} = \frac{a\left[n^2\left(1-(n-2)v\right)^2 + (1+v)^2\right]}{b\delta\left((2n+1-n^2)v+n+1\right)^2},$$

we thus have

$$\begin{aligned}\phi_i^*(S_2) - \phi_c^*(S_2) &= \frac{(1-(n-2)v)a}{((2n+1-n^2)v+n+1)b} - \frac{a-(XS_2+Y)}{(n+1)b} \\ &= \frac{av(n-1)\Gamma}{b\delta n(n+1)\left((2n+1-n^2)v+n+1\right)^2},\end{aligned}$$

where

$$\Gamma = n^2(\delta - r)(1-(n-2)v) - (2\delta - r + \delta n)(1+v).$$

Therefore, $\phi_i^*(S_2) - \phi_c^*(S_2)$ has the same sign as $\Gamma$. We can express $\Gamma$ as

$$\Gamma(n,v) \equiv \left[(r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta\right]v + (n+1)\big(r - 2\delta + n(\delta - r)\big),$$

where $\Gamma(n,v)$ is a linear function in $v$ with

$$\frac{\partial \Gamma(n,v)}{\partial v} = (r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta < 0,$$

since $2\delta - r > 0$ and $\delta - r > 0$.

Note that

$$\Gamma(n,0) = (n+1)\big(r - 2\delta + n(\delta - r)\big) = (n+1)\Big((n-2)\delta - (n-1)r\Big),$$

so when $n = 2$, $\Gamma(n,0) < 0$, and for $n \geq 3$, we have $\Gamma(n,0) > 0$ if $\delta > \frac{n-1}{n-2}r$, and $\Gamma(n,0) < 0$ if $\delta < \frac{n-1}{n-2}r$. But from Assumption 1 and condition (C4), we need to ensure that

$$\delta > \frac{(n^2+1)}{2}r.$$

It can be easily observed that for all $n \geq 3$,

$$\frac{(n^2+1)}{2} - \frac{n-1}{n-2} = \frac{n(n^2-2n-1)}{2(n-2)} > 0,$$

and thus

$$\delta > \frac{(n^2+1)}{2}r > \frac{n-1}{n-2}r.$$

Therefore, we have $\Gamma(n,0) < 0$ for $n = 2$ and $\Gamma(n,0) > 0$ for $n \geq 3$.

Also, note that

$$\begin{aligned}\Gamma(n, \frac{1}{n-1}) &= \left[(r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta\right]\frac{1}{n-1} + (n+1)\big(r - 2\delta + n(\delta - r)\big)\\ &= -\frac{n(2\delta - r + nr)}{n-1} < 0.\end{aligned}$$

Given the linearity of the function $\Gamma(n,v)$, we must have $\Gamma(n,v) < 0$ for all $n = 2$ and $v \in (0, \frac{1}{n-1})$, and when $n \geq 3$, there must exist some threshold shareholding $\hat{v}$ such that $\Gamma(n,v) > 0$ for any $v \in (0,\hat{v})$ and $\Gamma(n,v) < 0$ for any $v \in (\hat{v}, \frac{1}{n-1})$, where $\Gamma(n,\hat{v}) = 0$, i.e.,

$$\hat{v} = \frac{(n+1)\Big(r - 2\delta + n(\delta - r)\Big)}{(\delta - r)\big(n^2(n-2)\big) + 2\delta - r + n\delta} > 0.$$

We now show that $\hat{v}$ is less than the upper bound of shareholdings $\frac{1}{n-1}$ by direct comparison:

$$\hat{v} = \frac{(n+1)\big(r - 2\delta + n(\delta - r)\big)}{(\delta - r)\big(n^2(n-2)\big) + 2\delta - r + n\delta} < \frac{1}{n-1},$$

$$\iff (n^2-1)\big(r - 2\delta + n(\delta - r)\big) < (\delta - r)\big(n^2(n-2)\big) + 2\delta - r + n\delta,$$

$$\iff -n(2\delta - r + nr) < 0.$$

To conclude, we have the following cases:

1. If $n = 2$, then $\phi_i^*(S_2) < \phi_c^*(S_2)$ for all $v \in (0, \frac{1}{n-1})$;

2. If $n \geq 3$, then $\phi_i^*(S_2) < \phi_c^*(S_2)$ for $v \in (\hat{v}, \frac{1}{n-1})$, $\phi_i^*(S_2) = \phi_c^*(S_2)$ when $v = \hat{v}$, and $\phi_i^*(S_2) > \phi_c^*(S_2)$ for $v \in (0, \hat{v})$.

Given that $S_1 > S_{1,N}$, $S_2 < S_{2,N}$, $q_c > q_i^v$ and $\Omega_i > \Omega_c$ (from Lemma 1), we must have the following scenarios:

1. For any $n = k = 2$ and $v \in (0, \frac{1}{n-1})$, or $n = k \geq 3$ and $v \in [\hat{v}, \frac{1}{n-1})$, $\phi_c^*(S) \geq \phi_i^*(S)$;

2. For any $n = k \geq 3$ and $v \in (0, \hat{v})$, there exists a $\hat{S}_1 \in (S_1, S_2)$ and a $\hat{S}_2 \in (S_2, S_{2,N})$ such that $\phi_i^*(S) > \phi_c^*(S)$ if and only if $\hat{S}_1 < S < \hat{S}_2$.

□

# F Proof of Lemma 3

*Proof.* Note that

$$\frac{dQ_v}{dv} = -\frac{ak(k-1)}{b\big((k+n+1-k^2)v+n+1\big)^2} < 0, \quad \frac{d\zeta_v}{dv} = \frac{k(k-1)(2\delta-r)}{2\Big((k+n-k^2)v+n\Big)^2} > 0$$

Therefore, for any $n \geq k \geq 2$ and $v > 0$, we have

$$Q_v(v) < Q_v(0) = Q_c, \quad \zeta_v(v) > \zeta_v(0) = \zeta_c.$$

□

# G Proof of Proposition 4

*Proof.* Note that

$$\begin{aligned}
\Phi_v^*(S_2) - \Phi_c^*(S_2) &= \frac{((-k^2+n+k)v+n)a}{((k+n+1-k^2)v+n+1)b} - \frac{n(a-(XS_2+Y)}{(n+1)b} \\
&= \frac{akv(k-1)\Big[n(\delta-r)\Big(n(1+v)-k(k-1)v\Big)-(2\delta-r+\delta n)(1+v)\Big]}{b\delta n(n+1)\Big((k+n+1-k^2)v+n+1\Big)^2}.
\end{aligned}$$

Thus, for any $v \in (0, \frac{1}{k-1})$ and $n > k \geq 2$, $\Phi_v^*(S_2) - \Phi_c^*(S_2)$ has the same sign as

$$\begin{aligned}
\Theta(k,n,v) &\equiv n(\delta-r)\Big(n(1+v)-k(k-1)v\Big)-(2\delta-r+\delta n)(1+v) \\
&= \Big[(n+1)\Big((n-2)\delta-(n-1)r\Big)-k(k-1)n(\delta-r)\Big]v+(n+1)\Big((n-2)\delta-(n-1)r\Big),
\end{aligned}$$

which is linear in $v$. Notice that

$$\Theta(k,n,0) = (n+1)\Big((n-2)\delta-(n-1)r\Big),$$

therefore, we can rewrite $\Theta(k,n,v)$ as

$$\Theta(k,n,v) = \Big(\Theta(k,n,0)-k(k-1)n(\delta-r)\Big)v+\Theta(k,n,0),$$

and we have

$$\Theta(k,n,\frac{1}{k-1}) = \Big(\Theta(k,n,0) - k(k-1)n(\delta - r)\Big)\frac{1}{k-1} + \Theta(k,n,0)$$
$$= \frac{k}{k-1}\Big[\Big(n(n-k)-2\Big)\delta - \Big(n(n-k)+n-1\Big)r\Big].$$

From Assumption 1 and condition (C4), we need to ensure that

$$\delta > \frac{(n^2+1)}{2}r.$$

It can be easily shown that for all $n > k \geq 2$,

$$\frac{(n^2+1)}{2} - \frac{n-1}{n-2} = \frac{n(n^2-2n-1)}{2(n-2)} > 0,$$

and

$$\frac{(n^2+1)}{2} - \frac{n(n-k)+n-1}{n(n-k)-2} = \frac{n(n+1)\big((n-1)(n-k)-2\big)}{2\big(n(n-k)-2\big)} \geq 0.$$

Therefore, we have

$$\delta > \frac{(n^2+1)}{2}r > \frac{n-1}{n-2}r > r,$$

and

$$\delta > \frac{(n^2+1)}{2}r \geq \frac{n(n-k)+n-1}{n(n-k)-2}r > r,$$

which means that for any $n > k \geq 2$,

$$\Theta(k,n,0) > 0, \quad \Theta(k,n,\frac{1}{k-1}) \geq 0.$$

This combined with the fact that $\Theta(k,n,v)$ is linear in $v$ completes the proof that $\Theta(k,n,v) > 0$ and thus $\Phi_v^*(S_2) > \Phi_c^*(S_2)$ for all $n > k \geq 2$ and $v \in (0, \frac{1}{k-1})$. Given that $S_1 > S_{1,N}$ and $S_2 < S_{2,N}$ (from Lemma 2), and $Q_v < Q_c$ and $\zeta_v > \zeta_c$ (from Lemma 3), there must exist a $\tilde{S}_1 \in (S_1, S_2)$ and a $\tilde{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ for any $S \in (\tilde{S}_1, \tilde{S}_2)$. □

# H Proof of Lemma 4

*Proof.* The stationary resource stocks with cross-ownership are given by

$$S_1^\infty(v) = \frac{a\Big[(2\delta - r)(1+v)^2 - r\Big(k(k-1)v - n(1+v)\Big)^2\Big]}{b\delta\Big((k+n+1-k^2)v + n + 1\Big)\Big[(2\delta - r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big)\Big]}$$

$$S_2^\infty(v) = \frac{a\Big((k+n-k^2)v+n\Big)}{b\delta\Big((k+n+1-k^2)v+n+1\Big)}, \quad S_3^\infty(v) = 1 - \frac{a(1-S_y)\Big((k+n-k^2)v+n\Big)}{b\delta S_y\Big((k+n+1-k^2)v+n+1\Big)},$$

while those without cross-ownership are denoted by

$$S_{1,N}^\infty = S_1^\infty(0) = \frac{a(2\delta - r(1+n^2)}{b\delta(1+n)(2\delta - r(1+n))},$$

$$S_{2,N}^\infty = S_2^\infty(0) = \frac{an}{b\delta(1+n)}, \quad S_{3,N}^\infty = S_3^\infty(0) = 1 - \frac{an(1-S_y)}{b\delta S_y(1+n)}.$$

Then, we have

$$\frac{\partial S_1^\infty}{\partial v} = \frac{2ak(k-1)(\delta-r)\left[(2\delta-r)(1+v)^2 + r\Big(k(k-1)v - n(1+v)\Big)^2\right]}{b\delta\Big((k+n+1-k^2)v+n+1\Big)^2\left[(2\delta-r)(1+v) + r\Big(k(k-1)v - n(1+v)\Big)\right]^2} > 0$$

$$\frac{\partial S_2^\infty}{\partial v} = -\frac{ak(k-1)}{b\delta\Big((k+n+1-k^2)v+n+1\Big)^2} < 0$$

$$\frac{\partial S_3^\infty}{\partial v} = \frac{ak(k-1)(1-S_y)}{b\delta S_y\Big((k+n+1-k^2)v+n+1\Big)^2} > 0$$

Therefore, for any $v > 0$, $S_1^\infty(v) > S_1^\infty(0) = S_{1,N}^\infty$, $S_2^\infty(v) < S_2^\infty(0) = S_{2,N}^\infty$ and $S_3^\infty(v) < S_3^\infty(0) = S_{3,N}^\infty$. □

# I Proof of Proposition 7

*Proof.* We need to show that for all $\delta$ that satisfies Assumption 1, $\Delta PS(S_{1,N}^\infty, S_1^\infty) > 0$ and $\Delta PS(S_{1,N}^\infty, S_3^\infty) > 0$. First, we have

$$\begin{aligned}\Delta PS(S_{1,N}^\infty, S_1^\infty) &= \frac{\big(1-(k-1)v\big)\big((-k^2+n+k)v+n\big)}{1-(k-2)v} W(S_1^\infty) - nW_c(S_{1,N}^\infty) \\ &= \frac{2a^2k(k-1)v(\delta-r)\Omega_1\Omega_2}{br(n+1)^2(2\delta-r-nr)^2\big((k+n+1-k^2)v+n+1\big)^2\big((2\delta-r-nr)(1+v)+k(k-1)rv\big)^2},\end{aligned}$$

where

$$\begin{aligned}\Omega_1(v) &= (2\delta - r + n^2r)(1+v) - k(k-1)nrv \\ &= \Big(2\delta - r + n^2r - k(k-1)nr\Big)v + 2\delta - r + n^2r,\end{aligned}$$

$$\Omega_2 = k^2(k-1)^2v^2r(2\delta - r(1+n^2)) - 2knv(k-1)(1+v)\Big(2\delta(\delta-r) - r^2(n^2-1)\Big)$$
$$+ (1+v)^2(n^2-1)(2\delta - r + nr)(2\delta - r(1+n)).$$

Note that $\Omega_1(v)$ is a linear function in $v$ with

$$\Omega_1(0) = 2\delta - r + n^2r > 0,$$

$$\Omega_1(\frac{1}{k-1}) = \frac{(2\delta - r + n^2r)(k-1) - k(k-1)nr}{k-1} > 0$$
$$= \frac{2\delta(k-1) + \Big(n(n-k) - 1\Big)(k-1)r}{k-1} > 0,$$

and thus $\Omega_1(v) > 0$ for all $v \in (0, \frac{1}{k-1})$. We also have

$$\Omega_2 = k^2(k-1)^2v^2r(2\delta - r(1+n^2)) - 2knv(k-1)(1+v)\Big(2\delta(\delta-r) - r^2(n^2-1)\Big)$$
$$+ (1+v)^2(n^2-1)(2\delta - r + nr)(2\delta - r(1+n))$$
$$= k(k-1)v\Big[4n(1+v)\delta(\delta-r) - 2k(k-1)\delta rv + \Big(k(k-1)(n^2+1)v - 2n(1+v)(n^2-1)\Big)r^2\Big]$$
$$+ (1+v)^2(n^2-1)(2\delta - r + nr)(2\delta - r(1+n)) > 0.$$

Therefore, we must have

$$\Delta PS(S^{\infty}_{1,N}, S^{\infty}_1) > 0, \forall\, \delta > \frac{(n^2+1)}{2}r.$$

Next, notice that

$$\Delta PS(S^{\infty}_{1,N}, S^{\infty}_3) = \frac{k\pi^v_i + (n-k)\pi^v_o}{r} - nW_c(S^{\infty}_{1,N})$$
$$= \frac{a^2\Lambda_1\Lambda_2}{br(n+1)^2(2\delta - r - nr)^2\Big((k+n+1-k^2)v + n + 1\Big)^2},$$

where

$$\Lambda_1 = k(k-1)v\big(2\delta - r(1+n^2)\big) + nr(n^2-1)(1+v) > 0,$$

and

$$\Lambda_2(v) = \Big((n^2-1)(2\delta - r) - 2kn(\delta - r)(k-1)\Big)v + (n^2-1)(2\delta - r),$$

which is a linear function in $v$ with

$$\Lambda_2(0) = (n^2 - 1)(2\delta - r) > 0$$

and

$$\begin{aligned}
\Lambda_2(\frac{1}{k-1}) &= \frac{k}{k-1}\Big((2\delta - r)(n^2 - 1) - 2n(\delta - r)(k-1)\Big) \\
&= \frac{k}{k-1}\Big[\Big(2n(n-k) + 2(n-1)\Big)\delta - \Big((n^2-1) - 2n(k-1)\Big)r\Big] \\
&> \frac{k}{k-1}\Big[\Big(2n(n-k) + 2(n-1)\Big)\frac{(n^2+1)}{2}r - \Big((n^2-1) - 2n(k-1)\Big)r\Big] \\
&= \frac{knr}{k-1}(n^2-1)(n-k+1) > 0.
\end{aligned}$$

Then, we must have $\Lambda_2(v) > 0$ for all $v \in (0, \frac{1}{k-1})$ and thus

$$\Delta PS(S^{\infty}_{1,N}, S^{\infty}_{3}) > 0, \forall\, \delta > \frac{(n^2+1)}{2}r.$$

□