# On cross-ownership in a nonrenewable resource differentiated product oligopoly*

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**Abstract**

We show that a symmetric cross-ownership is always profitable and is increasingly profitable in either levels or ratios, and it is most beneficial when the degree of competition takes on intermediate values. These static results also carry over to the case of a dynamic nonrenewable differentiated oligopoly. However, the dynamic case exhibits lower profitability when the initial resource stock owned by each differentiated firm is small enough, which is in sharp contrast with the games in quantity setting. Finally, we illustrate that the relative loss in welfare from cross-ownership may be a decreasing function of resource scarcity.

**Keywords:** Cross-ownership, Differentiated Products, Bertrand Competition, Oligopoly, Shareholdings, Nonrenewable Resources, Resource Stock
**JEL Codes:** L13, L41, Q3

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# 1 Introduction

In this paper, we examine whether the conclusions reached in the static benchmark can be extended to the case of differentiated nonrenewable resource oligopolies when rival firms compete *à* la Bertrand and engage in cross-shareholdings. Instead of product homogeneity, we assume that different varieties of similar products exist in the market, which is a more accurate reflection of reality. Moreover, there are numerous industries in which firms compete in a way that is more consistent with Bertrand competition. Using models with quantity competition to investigate price competition would often end up with unreliable results and give misleading policy implications. For example, Colombo and Labrecciosa (2015) find that the traditional result that the Bertrand equilibrium is more efficient than the Cournot equilibrium does not necessarily hold in a dynamic oligopolistic model of exploitation of a renewable productive asset.

Indeed, firms seldom sell exactly the same products. Take the global oil market as an example; there are different types of crude oil, e.g., Brent, West Texas Intermediate (WTI), Western Canadian Select (WCS), Dubai Fateh, Murban, Urals, and their prices differ substantially due to quality, regional infrastructure, and geopolitical events.[1] Brent and WTI often command higher prices due to their lighter, sweeter crude quality, while WCS is a heavier, sour crude with higher sulphur content, resulting in a relatively lower price than the two.[2] Meanwhile, Dubai Fateh and Murban are key benchmarks for Middle Eastern oil, with Dubai Fateh a medium-sour crude oil and Murban being a light crude, prices of which are highly affected by Asian demand, shipping costs, and OPEC production levels.[3] Finally, Urals is a heavy, sour crude oil blend from Russia. Its higher sulphur content and heavier nature, combined with the western sanctions, have made it trade at a huge discount compared to other oils.[4] In the case of coal, high-rank varieties such as anthracite and metallurgical bituminous ones often command premium prices than those lower ranked lignite and sub-bituminous grades, due to their high energy density and industrial applications (IEA, 2023). In another example, the critical metal nickel – essential for battery production in the energy transition – exhibits significant product heterogeneity, as nickel ores exist in two types of deposits: sulphide and laterite (IRENA, 2023). The former deposits are mainly found in Australia, Canada and Russia, and contain higher-grade nickel that can be more easily processed into Class 1 battery-grade nickel, while the latter ones are mainly produced from Indonesia and the Philippines, which contain relatively lower-grade nickel that requires additional energy-intensive processing for conversion

[1]See https://oilprice.com/oil-price-charts/ for prices across different types of crude oils.

[2]See https://www.oilsandsmagazine.com/technical/western-canadian-select-wcs/.

[3]See https://www.oxfordenergy.org/publications/dubai-we-have-a-problem-murban-and-middle-east-crude-pricing/ and https://www.reuters.com/business/energy/murban-crude-prices-drop-opec-raises-output-prompting-surge-volumes-asia-2025-05-23/.

[4]https://www.reuters.com/markets/commodities/russian-urals-oil-prices-fell-lowest-level-since-2023-brent-price-collapsed-2025-04-07/ and https://energyandcleanair.org/june-2025-monthly-analysis-of-russian-fossil-fuel-exports-and-sanctions/.

into battery-grade nickel (Paraskova, 2022). These low- and higher-grade nickel ores act as imperfect substitutes, competing in distinct yet interconnected markets (Kooroshy, Preston and Bradley, 2014).

Taking into account this specific feature in exhaustible industries, together with the widely existing cross-shareholding activities among mining firms (Kumar, 2012; Benchekroun, Breton and Chaudhuri, 2019; Benchekroun, Dai and Long, 2022), we introduce product differentiation à la Dai, Benchekroun and Long (2022) and investigate the impact of cross-ownership in a differentiated product oligopoly. We first characterize the static and dynamic equilibrium outcomes under a $k-$symmetric cross-ownership structure in which a subset of $k \leq n$ firms engage in rival cross-shareholdings and each firm has an equal silent financial interest in the other firms, while the remaining $n-k$ firms stay independent.

We show that in a static model, when firms engage in price competition in an oligopolistic industry consisting of symmetrically differentiated products, rival cross-shareholdings will increase both cross-ownership participants and non-participants' product prices, but cross-owners will reduce their output while outsiders take a free ride by expanding their production. These results are consistent with the predictions from the static oligopoly and cross-ownership theory, which conceptualize cross-ownership as a form of "partial merger" that entails anti-competitive effects (Reynolds and Snapp, 1986; Bresnahan and Salop, 1986; Farrell and Shapiro, 1990; Flath, 1991, 1992; O'Brien and Salop, 2000; Dietzenbacher, Smid and Volkerink, 2000; Brito, Cabral and Vasconcelos, 2014; Brito, Ribeiro and Vasconcelos, 2014; Brito et al., 2018; Benndorf and Odenkirchen, 2021; Hariskos, Königstein and Papadopoulos, 2022).

Further, using a dynamic game model in which firms compete *à* la Bertrand while each differentiated firm faces a resource stock constraint, i.e., the output of each resource-extracting firm is constrained by its limited initial resource stock, we characterize an open-loop Nash-Bertrand cross-ownership equilibrium (OL-NBCOE) of the game. We find that under the $k$-symmetric cross-ownership structure, outsiders start with a higher exploitation rate but charge a lower product price compared to the cross-owners. As more resources gets depleted, this trend reverses, and eventually, the outsiders will exhaust their stocks earlier than the cross-ownership participants at any resource stock level, leading to increasingly concentrated supply over time. This result resembles the 'oil'igopoly theory (Loury, 1986; Polasky, 1992), which predicts that small firms will deplete their reserves before large ones do, potentially leading to eventual market monopolization. The increased concentration over time induced by rival cross-shareholdings confers market power on those cross-ownership participants. As such, the cross-owners can raise prices more than in other industries without stock constraints.

Next, we move to investigate the profitability of cross-ownership in the differentiated product oligopoly. Our results indicate that the static profitability is always positive, irrespective of the degree of cross-ownership or the intensity of market competition. In addition, cross-ownership is increasingly profitable in either levels or ratios, and it is most beneficial when the degree of competition takes on intermediate values. This result bears some similarity with the merger result in Deneckere and Davidson (1985). Moreover, we find that these static results also carry over to the case of a dynamic nonrenewable differentiated oligopoly. However, it can be shown that when the initial resource stock owned by each firm is small enough, a $k$-symmetric cross-ownership yields less profit gains in the case of a differentiated nonrenewable resource oligopoly than other industries without stock constraints. This result sharply contrasts with the typical conclusion obtained in the resource game under Cournot competition, where the presence of resource scarcity makes it more likely to be profitable even under the scenario for which it is strictly unprofitable in the corresponding static framework (Benchekroun, Breton and Chaudhuri, 2019; Dai, Benchekroun and Long, 2022).[5] Unlike in the quantity competition setting, where outsiders are more constrained in their response to any output reduction brought by insiders due to their limited resource stocks, the resource constraints seem to work in the opposite direction here by inducing the outsiders to aggressively lower their product price and expand their production under price competition. Consequently, this intensified competition reduces the profitability of cross-ownership in a differentiated nonrenewable resource oligopoly when the resource stock is small.

Finally, we compare the static and dynamic welfare effects of cross-ownership. While a $k$-symmetric cross-ownership in a differentiated Bertrand oligopoly is never welfare-improving in both cases, we find that the welfare loss can be smaller in the dynamic case than in the static one, provided that the initial resource stock owned by each differentiated firm is small enough. This occurs because after the outsiders exhaust their resource stocks, the market will be monopolized by the group of cross-owners, who can then raise their product prices much higher. While higher prices are detrimental to consumer surplus, they nevertheless extend the duration over which the resources can be exploited. As resources become increasingly scarce, the availability and extended periods of use of these resources partially offset the negative effect of higher prices on the consumer surplus. Consequently, the smaller loss in consumer surplus due to increased scarcity and the increased profits due to higher prices will result in a smaller welfare loss in the case of a differentiated nonrenewable resource oligopoly than that in the static case.

The remainder of the paper is structured as follows. Section 2 first presents the static model in a differentiated product oligopoly and then the dynamic model of a differentiated nonrenewable resource industry. Section 3 conducts the profitability

[5] Similarly, in the dynamic games of exploitation of common-pool renewable resources, Benchekroun and Gaudet (2015); Colombo and Labrecciosa (2018); Dai, Benchekroun and Dahmouni (2024) show that industry output and consumer surplus can increase following full or partial cooperation among resource users. Meanwhile, Benchekroun, Chaudhuri and Tasneem (2020) demonstrate that Free Trade may lead to a lower discounted sum of consumer surplus and of social welfare than Autarky. These surprising results, obtained within the dynamic Cournot oligopoly framework, sharply contrast with the static outcomes.

analysis of cross-ownership. Section 4 provides a welfare analysis. Finally, Section 5 concludes with the summary of our findings.

# 2 Bertrand competition in a differentiated oligopoly

## 2.1 The static model

We consider an oligopolistic industry consisting of $n$ symmetrically differentiated products, each produced by a separate firm. Denote the set of firms as $J = \{1, 2, \cdots, n\}$, indexed by $j$. Firms compete *à* la Bertrand and own the exclusive technology for the production of their particular product. Marginal costs are constant and identical across all firms, assumed at $c = 0$ for simplicity. Suppose the demand for firm $j$'s product is a function of the prices $(p_1, p_2, \cdots, p_n)$. There is a numeraire good, called good $q_0$, in the background, and $p_0 = 1$. Following Shubik and Levitan (2013) and Singh and Vives (1984), we specify the utility function as linear in good $q_0$ and quadratic in good $(q_1, q_2, \cdots, q_n)$ :

$$U(q_0, q_1, q_2, \cdots, q_n) = q_0 + V \sum_{j=1}^{n} q_j - \frac{n+\gamma}{2n(1+\gamma)} \sum_{j=1}^{n} q_j^2 - \frac{\gamma}{n(1+\gamma)} \sum_{j \neq l} q_l q_j,$$

where $V$ is a positive constant, and $\gamma \geq 0$ is a substitutability parameter representing the product differentiation across goods. Without loss of generality, we set $V = 1$ in what follows. As $\gamma$ increases, product differentiation decreases. When $\gamma$ approaches zero, goods become unrelated, and each firm becomes a monopoly of its specific product. When $\gamma$ approaches infinity, goods become perfect substitutes. The consumer surplus can thus be defined as the utility minus the payments for the goods:

$$CS = U(q_0, q_1, q_2, \cdots, q_n) - \sum_{j=0}^{n} p_j q_j.$$

Then, the demand function for firm $j$ is given by

$$q_j(p_1, p_2, \cdots, p_n) = 1 - p_j - \gamma \left( p_j - \frac{1}{n} \sum_{l=1}^{n} p_l \right), \quad j, l \in J.$$

Following Dai, Benchekroun and Long (2022), we consider a situation in which a subset of $k$ firms ($2 \leq k \leq n$) engage in rival cross-shareholdings and each firm has an equal ownership stake $v$ in the other firms, while the remaining $n - k$ firms stay independent. We use the subsets $I = \{1, 2, \cdots, k\}$, indexed by $i$ and $O = \{k + 1, \cdots, n\}$, indexed by $o$, referring to the insiders and outsiders to the cross-ownership, respectively. In an industry characterized by rival cross-shareholdings, the aggregate profits of firm $j$ consist not only of its own operating profits but also a share of profits in the competitors through its direct and indirect ownership links (Flath, 1992; Gilo, Moshe and Spiegel, 2006). Then, firm $j$'s problem can be expressed as

$$\max_{p_j \geq 0} \Pi_j = \pi_j + v \sum_{i \neq j} \Pi_i = p_j q_j(p_1, p_2, \cdots, p_n) + v \sum_{i \neq j} \Pi_i,$$

where $\pi_j = p_j q_j(p_1, p_2, \cdots, p_n)$ denotes firm $j$'s operating profits and $v$ represents firm $j$'s fractional shareholdings in firm $i$ for any $i \neq j$. We make the following assumption:

**Assumption 1.** *Each firm seeks to maximize the value of its aggregate profits, but controls only its own price decision $p_j$, with rival shareholdings $0 < v < \frac{1}{k-1}$, i.e., firms only have a silent financial interest or non-controlling minority stake in the rivals.*

Assumption 1 guarantees that the aggregate stake of rivals in each cross-ownership participant $(k-1)v$ is less than 1. Let $\boldsymbol{\Pi}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ denote the $n \times 1$ vectors of aggregate profits, prices and outputs, respectively, and $\boldsymbol{D}$ denote the $n \times n$ cross-shareholding matrix, then the aggregate profit functions can be expressed in matrix form as

$$\boldsymbol{\Pi} = \boldsymbol{p}\boldsymbol{q}(\boldsymbol{p}) + \boldsymbol{D}\boldsymbol{\Pi},$$

where $\boldsymbol{D} = \begin{bmatrix} \boldsymbol{A}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-k} \end{bmatrix}$, and $\boldsymbol{A}_{kk}$ is a $k \times k$ matrix with element 0 in the diagonal and $v$ off-diagonal. This set of $n$ equations implicitly defines the aggregate profits for each differentiated firm. Then $\boldsymbol{I} - \boldsymbol{D} = \begin{bmatrix} \boldsymbol{B}_{kk} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix}$, where $\boldsymbol{B}_{kk}$ is a $k \times k$ matrix with element 1 in the diagonal and $-v$ off-diagonal, and $\boldsymbol{I}_{n-k}$ denote the $(n-k) \times (n-k)$ identity matrix.

Under Assumption 1, matrix $\boldsymbol{I} - \boldsymbol{D}$ is invertible and thus the aggregate profit function for each differentiated firm can be solved as:

$$\boldsymbol{\Pi} = (\boldsymbol{I} - \boldsymbol{D})^{-1} \boldsymbol{p}\boldsymbol{q}(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{B}_{kk}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}\boldsymbol{q}(\boldsymbol{p}) \end{bmatrix},$$

where $\boldsymbol{B}_{kk}^{-1}$ is given by the following matrix

$$\Omega \equiv \frac{1}{f(v)} \begin{bmatrix} 1-(k-2)v & v & \cdots & v \\ v & 1-(k-2)v & \cdots & v \\ \vdots & \vdots & \ddots & \vdots \\ v & v & \cdots & 1-(k-2)v \end{bmatrix},$$

with $f(v) = (1+v)(1-(k-1)v) > 0$. The aggregate profit function of firm $i \in I$ is thus given by

$$\Pi_i = \frac{1}{f(v)} \left[ (1-(k-2)v)\pi_i + v \sum_{m \in I \setminus i} \pi_m \right] = \frac{1}{f(v)} \left[ (1-(k-2)v) p_i q_i + v \sum_{m \in I \setminus i} p_m q_m \right],$$

while for firm $o \in O$, the aggregate profit function is

$$\Pi_o = \pi_o = p_o q_o.$$

Firm $j$ takes other firms' prices ($p_{-j}$) as given and chooses $p_j$ to maximize its aggregate profit. The first-order conditions yield

$$\Big(1-(k-2)v\Big)\left[1-p_i-\gamma\left(p_i-\frac{1}{n}\sum_{j=i}^{n}p_j\right)-p_i\left(1+\gamma(1-\frac{1}{n})\right)\right]+v\sum_{m\in I\setminus i}p_m\left(\frac{\gamma}{n}\right)=0,$$

$$1-p_o-\gamma\left(p_o-\frac{1}{n}\sum_{j=1}^{n}p_j\right)-p_o\left(1+\gamma(1-\frac{1}{n})\right)=0.$$

Exploiting symmetry, the interior solution yields the static Bertrand equilibrium prices with cross-ownership as

$$p_i^v = \frac{n\big(1-(k-2)v\big)\big(2n+(2n-1)\gamma\big)}{F(v)}, \tag{1}$$

$$p_o^v = \frac{n\big(2n\big(1-(k-2)v\big)(1+\gamma)-(1+v)\gamma\big)}{F(v)}, \tag{2}$$

and thus the corresponding equilibrium quantities are

$$q_i^v = \frac{\big(2n+(2n-1)\gamma\big)\big(\big(1-(k-2)v\big)(1+\gamma)n-(1+v)\gamma\big)}{F(v)}, \tag{3}$$

$$q_o^v = \frac{\big(n+(n-1)\gamma\big)\big(2n\big(1-(k-2)v\big)(1+\gamma)-(1+v)\gamma\big)}{F(v)}, \tag{4}$$

where $F(v) = A + Bv > 0$,[6] with

$$A = \big(2n+(2n-1)\gamma\big)\big(2n+(n-1)\gamma\big) > 0,$$
$$B = -\big((n-1)(2n(k-2)+1)+k(k-1)\big)\gamma^2 - 2n\big(k(3n-1)-3(2n-1)\big)\gamma - 4n^2(k-2).$$

As a comparison, the standard equilibrium Bertrand price and quantity without cross-ownership are obtained by setting $v = 0$:

$$p_b = \frac{n}{2n+(n-1)\gamma}, \quad q_b = \frac{n+(n-1)\gamma}{2n+(n-1)\gamma}. \tag{5}$$

Then, we can easily establish the following result:

**Proposition 1.** *For any $\gamma \geq 0$, $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$,*

*(i) $p_i^v > p_o^v > p_b$ and $q_o^b > q_b > q_i^b$;*

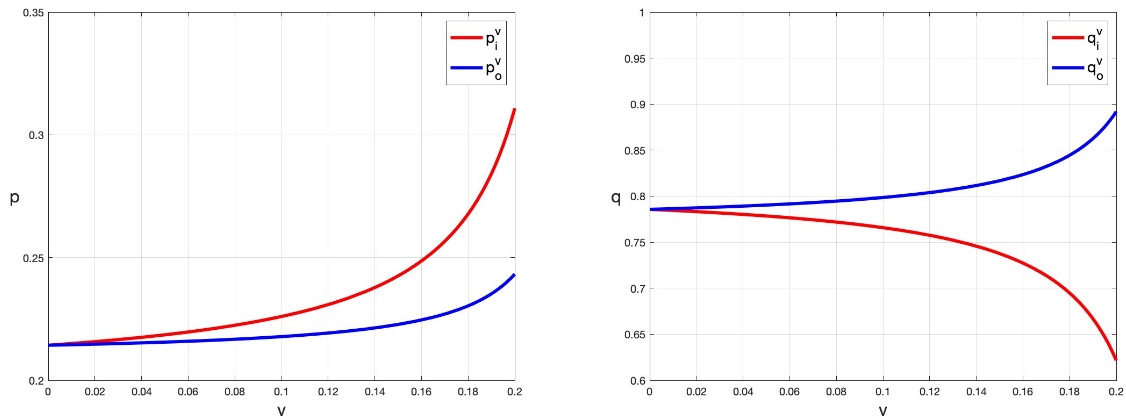
*(ii) $p_i^v, p_o^v$ and $q_o^v$ increase in $v$, but $q_i^v$ decreases in $v$.*

[6] See Appendix A for the proof.

*Proof.* See Appendix A. □

A symmetric cross-ownership between a subset of firms in the industry increases both the insiders' and outsiders' product prices over the standard Bertrand one, but the magnitude is larger for insiders than for outsiders. At the same time, cross-owners reduce their quantities, but outsiders expand their production, following the rival cross-shareholdings. These results are highly intuitive. Indeed, when firms have an ownership stake in their rivals, they will have an incentive to compete less aggressively and thus unilaterally increase their prices and reduce their quantities, as one firm's gain may come at the loss of the other firms in which it has financial interests. But in terms of strategic complements in Bertrand competition, the outsiders will take advantage of the price increase and quantity reduction brought by these cross-owners, raising their product prices (but slightly below the insiders' prices) and expanding their production in order to steal the market share from their competitors.

Moreover, an increase in the level of cross-ownership ($v$) will result in less competition among insiders, thus inducing them to increase their prices and reduce outputs by more. Meanwhile, the stronger free-riding incentives will prompt outsiders to respond more aggressively by raising their product prices and expanding their production further. This result is confirmed in Figure 1, where the equilibrium insiders' and outsiders' prices $(p_i^v, p_o^v)$ and quantities $(q_i^v, q_o^v)$ are plotted as a function of $v$ when $\gamma = 3, k = 6, n = 9$.



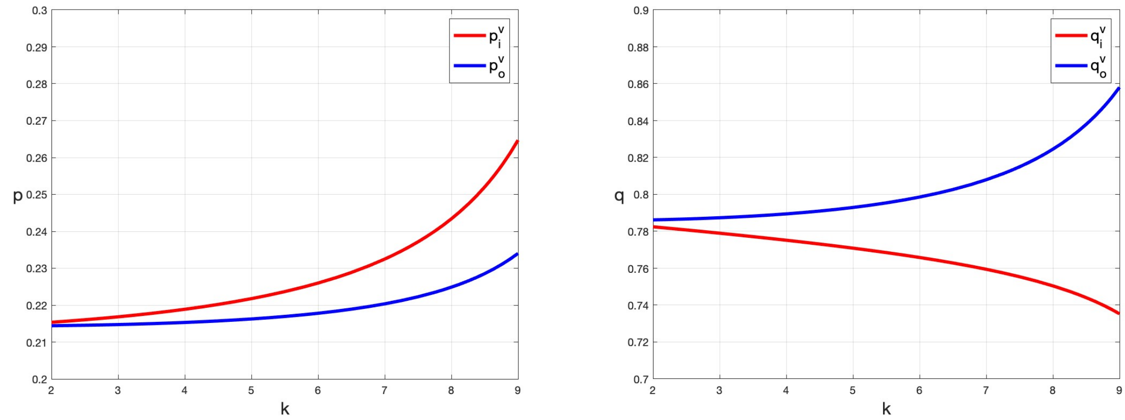

(a) Prices as a function $v$

(b) Quantities as a function $v$

Figure 1: The static Bertrand equilibrium with cross-ownership as a function of $v$

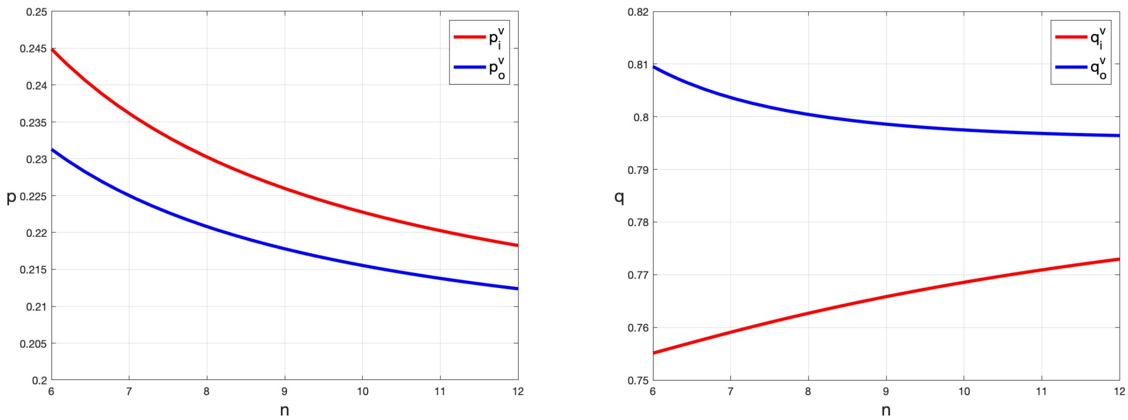
Furthermore, we conduct a comparative statics analysis of the static Bertrand equilibrium under the $k-$symmetric cross-ownership structure with respect to $k, n$ and $\gamma$. We plot the equilibrium prices $(p_i^v, p_o^v)$ and quantities $(q_i^v, q_o^v)$ as a function of $k$ when $\gamma = 3, n = 9, v = 0.1$ in Figure 2, of $n$ when $\gamma = 3, k = 6, v = 0.1$ in Figure 3, and of $\gamma$ when $k = 6, n = 9, v = 0.1$ in Figure 4, respectively. We observe the following result:

**Result 1.** *The equilibrium Bertrand prices with cross-ownership $(p_i^v, p_o^v)$ are increasing in $k$ but decreasing in $n$ or $\gamma$, and the equilibrium output of cross-owners $(q_i^v)$ is decreasing in $k$ but* *increasing in $n$ and $\gamma$, while the outsiders' output is increasing in $k$ and $\gamma$ but decreasing in $n$.*



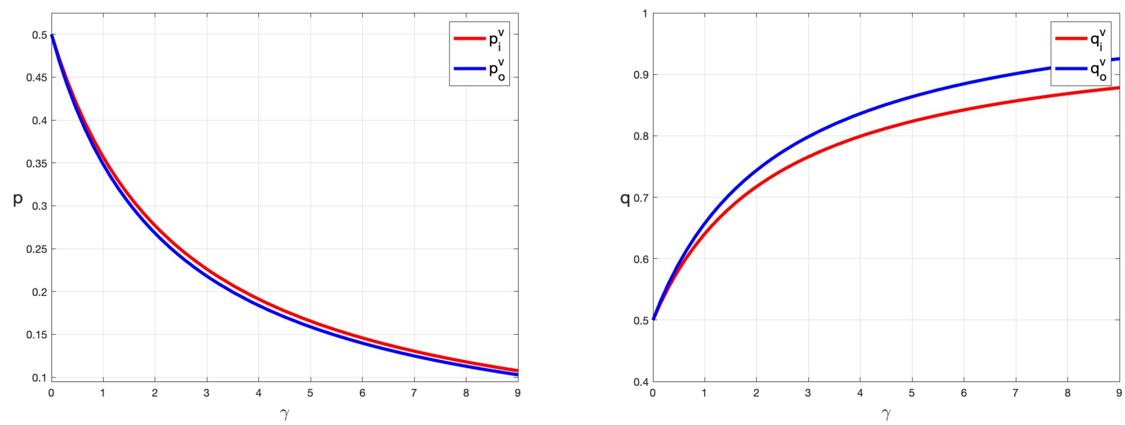

(a) Prices as a function $k$ (b) Quantities as a function $k$

Figure 2: The static Bertrand equilibrium with cross-ownership as a function of $k$




(a) Prices as a function $n$ (b) Quantities as a function $n$

Figure 3: The static Bertrand equilibrium with cross-ownership as a function of $n$




(a) Prices as a function $\gamma$ (b) Quantities as a function $\gamma$

Figure 4: The static Bertrand equilibrium with cross-ownership as a function of $\gamma$

Notice that an increase in $k$ or a decrease in $n$ has the same effects as an increase in $v$ on the equilibrium product prices and outputs, as changes in the cross-ownership structure can either come from a change in the ownership level $v$ or ratios $k/n$ – the number of firms in the whole industry that participate in cross-shareholdings. Meanwhile, an increase in the substitutability parameter $\gamma$ (or a decrease in product differentiation) intensifies competition between firms, resulting in lower prices but higher quantities for both insiders and outsiders. Simulations using any other combinations of $k, n, v, \gamma$ show that Result 1 is qualitatively robust.

## 2.2 The dynamic model

We now consider the case of a nonrenewable resource industry that involves $n$ firms, each producing a differentiated product with the same initial stock endowments $S_{0j} = S$ and the same marginal cost of production assumed at 0. Firms are oligopolists in the resource market where they compete *à* la Bertrand. Let $p_j(t) \geq 0$ denote the price and $q_j(t) \geq 0$ denote the extraction rate for firm $j$'s product at time $t$. Demand for firm $j$'s resource is stationary and given by

$$q_j(p_1(t), p_2(t), \cdots, p_n(t)) = 1 - p_j(t) - \gamma \left( p_j(t) - \frac{1}{n} \sum_{l=1}^{n} p_l(t) \right), \quad j, l \in J.$$

Then, the aggregate profits of firm $j$ at time $t$ is as follows:

$$\Pi_j(t) = \pi_j(t) + v \sum_{i \neq j} \Pi_i(t) = p_j(t) q_j(t) + v \sum_{i \neq j} \Pi_i(t).$$

Each firm $j$ takes the price paths of all other firms $p_{-j}(t)$ as given and chooses its own price path $p_j(t)$ to maximize the discounted sum of the aggregate profits, which consists of its operating profit and the share of profits obtained through ownership interests in other firms, subject to its resource constraint:

$$\max_{p_j(t) \geq 0} \int_0^\infty e^{-\rho t} \left[ p_j(t) q_j(t) + v \sum_{i \neq j} \Pi_i(t) \right] dt,$$

$$s.t. \quad \int_0^\infty q_j(t) dt \leq S_{0j}.$$

Under the assumption of $0 < v < \frac{1}{k-1}$, it is possible to solve for the aggregate profit equation at each time $t$, and thus the problem of all firms can be reformulated as

$$\max_{\boldsymbol{p}(t) \geq 0} \int_0^\infty e^{-\rho t} \left( \begin{bmatrix} \boldsymbol{B}_{kk}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-k} \end{bmatrix} \boldsymbol{p}(t) \boldsymbol{q}(\boldsymbol{p}(t)) \right) dt,$$

$$s.t. \quad \int_0^\infty \boldsymbol{q}(t) dt \leq \boldsymbol{S_0}(t) dt,$$

where $\boldsymbol{S_0} = [S_{01}, S_{02}, \cdots, S_{0n}]'$. Then for a typical firm $i \in I$,

$$\max_{p_i(t)\geq 0} \int_0^\infty e^{-\rho t}\left[\frac{1}{f(v)}\Big((1-(k-2)v)\,p_i(t)q_i(t) + v\sum_{m\in I\setminus i} p_m(t)q_m(t)\Big)\right]dt,$$

$$s.t. \quad \int_0^\infty q_i(t)dt \leq S_{0i},$$

while for a typical firm $o \in O$,

$$\max_{p_o(t)\geq 0} \int_0^\infty e^{-\rho t}\Big[p_o(t)q_o(t)\Big]dt,$$

$$s.t. \quad \int_0^\infty q_o(t)dt \leq S_{0o}.$$

We characterize an open-loop Nash-differentiated Bertrand cross-ownership equilibrium (OL-NBCOE) of this game. More precisely,

**Definition 1** (Open-loop Nash-differentiated Bertrand Cross-ownership Equilibrium (OL-NBCOE))**.** *An n-tuple vector of price paths* $\boldsymbol{p} = (p_1, p_2, \cdots, p_k, p_{k+1}, \cdots, p_n)$ *with* $p(t) \geq 0$ *for all* $t \geq 0$ *is an open-loop Nash-differentiated Bertrand cross-ownership equilibrium if*

*(i) every price path is admissible and satisfies the corresponding resource constraint,*

*(ii) for all* $i \in I$,

$$\int_0^\infty e^{-\rho t}\left[\frac{1}{f(v)}\Big((1-(k-2)v)\,p_i(t)q_i(t) + v\sum_{m\in I\setminus i} p_m(t)q_m(t)\Big)\right]dt$$
$$\geq \int_0^\infty e^{-\rho t}\left[\frac{1}{f(v)}\Big((1-(k-2)v)\,p_l(t)q_l(t) + v\sum_{m\in I\setminus l} p_m(t)q_m(t)\Big)\right]dt$$

*for all* $p_l$ *satisfying the resource constraint, and*

*(iii) for all* $o \in O$,

$$\int_0^\infty e^{-\rho t}\Big[p_o(t)q_o(t)\Big]dt \geq \int_0^\infty e^{-\rho t}\Big[p_m(t)q_m(t)\Big]dt$$

*for all* $p_m$ *satisfying the resource constraint.*

We now proceed to characterize an OL-NBCOE of the above-defined game. Let $T_i$ and $T_o$ denote the time at which firm $i \in I$ and firm $o \in O$ deplete their stocks, and denote by $p_i$ and $p_o$ the price paths and by $q_i$ and $q_o$ the extraction paths of firm $i \in I$ and firm $o \in O$, respectively. Then,

**Proposition 2.** *Assume that the initial stocks of all firms are equal, i.e., $S_{0j} = S$, and let*

$$p_i(t) = \begin{cases} \frac{n\left(1-(k-2)v\right)\left(2n(1+\gamma)-\gamma\right)+X\lambda_i+Y\lambda_o}{A+Bv} & \text{for } 0 \le t \le T_o \\ \frac{\left(1-(k-2)v\right)(1+\gamma)n^2+\left[\left(1-(k-2)v\right)(1+\gamma)n-(1+v)\gamma\right](n+k\gamma)e^{\rho(t-T_i)}}{\left(1-(k-2)v\right)(1+\gamma)(2n^2+k\gamma n)-(1+v)(n\gamma+k\gamma^2)} & \text{for } T_o \le t \le T_i\,, \\ 1 & \text{for } t \ge T_i \end{cases} \tag{6}$$

$$p_o(t) = \begin{cases} \frac{n\left(\left(1-(k-2)v\right)2n(1+\gamma)-(1+v)\gamma\right)+Z\lambda_i+\Gamma\lambda_o}{A+Bv} & \text{for } 0 \le t \le T_o \\ \frac{2\left(1-(k-2)v\right)(1+\gamma)n^2-(1+v)n\gamma+\left[\left(1-(k-2)v\right)(1+\gamma)n-(1+v)\gamma\right]k\gamma e^{\rho(t-T_i)}}{\left(1-(k-2)v\right)(1+\gamma)(2n^2+k\gamma n)-(1+v)(n\gamma+k\gamma^2)} & \text{for } T_o \le t \le T_i\,, \\ 1 & \text{for } t \ge T_i \end{cases} \tag{7}$$

*where*

$$\begin{aligned}
X &= \left(1-(k-2)v-(k-1)v^2\right)\left(n(1+\gamma)-\gamma\right)\left((n+k-1)\gamma+2n\right), \\
Y &= \left(1-(k-2)v\right)\left(n(1+\gamma)-\gamma\right)(n-k)\gamma, \\
Z &= \left(1-(k-2)v-(k-1)v^2\right)\left(n(1+\gamma)-\gamma\right)k\gamma, \\
\Gamma &= \left(\left(1-(k-2)v\right)\left(2n(1+\gamma)-k\gamma\right)-(1+v)\gamma\right)\left(n(1+\gamma)-\gamma\right), \\
\lambda_i &= \frac{\left[\left(1-(k-2)v\right)(1+\gamma)n-(1+v)\gamma\right]}{\left(1-(k-2)v-(k-1)v^2\right)\left(n(1+\gamma)-\gamma\right)}e^{\rho(t-T_i)}, \\
\lambda_o &= \frac{k\gamma\left[(1+v)\gamma-\left(1-(k-2)v\right)(1+\gamma)n\right]e^{\rho(t-T_i)}+n\left[(1+v)\gamma-2n\left(1-(k-2)v\right)(1+\gamma)\right]e^{\rho(t-T_o)}}{\left[(1+v)(n+k\gamma)\gamma-\left(1-(k-2)v\right)(1+\gamma)(2n+k\gamma)n\right]}.
\end{aligned}$$

*Then, the n-tuple vector $\boldsymbol{p}^{eq}$ where $p_j^{eq} = p_i$ when $j = 1, 2, \cdots, k$ and $p_j^{eq} = p_o$ when $j = k+1, \cdots, n$ constitutes an OL-NBCOE, and the equilibrium extraction paths are given by*

$$q_i(t) = 1-(1+\gamma)p_i(t)+\frac{\gamma}{n}\Big(kp_i(t)+(n-k)p_o(t)\Big), \tag{8}$$

$$q_o(t) = 1-(1+\gamma)p_o(t)+\frac{\gamma}{n}\Big(kp_i(t)+(n-k)p_o(t)\Big), \tag{9}$$

*with $T_i$ and $T_o$ the unique solutions to*

$$\int_0^{T_i} q_i(t)dt = \int_0^{T_i}\left[1-(1+\gamma)p_i(t)+\frac{\gamma}{n}\Big(kp_i(t)+(n-k)p_o(t)\Big)\right] = S, \tag{10}$$

$$\int_0^{T_o} q_o(t)dt = \int_0^{T_o}\left[1-(1+\gamma)p_o(t)+\frac{\gamma}{n}\Big(kp_i(t)+(n-k)p_o(t)\Big)\right] = S. \tag{11}$$

*Proof.* See Appendix B. □

Proposition 2 shows that given an initial resource stock $S$, all firms will exhaust their resource stocks in a finite time. Moreover, the outsiders will deplete their stocks earlier than the insiders, i.e., $T_i > T_o$ for all $S \ge 0$, $\gamma \ge 0$ and $0 < v < \frac{1}{k-1}$. This is in line with both the cross-ownership theory and standard oligopoly theory. When a firm acquires a partial ownership stake in a rival, it tends to compete less aggressively and thus unilaterally increases its product price. As a best response in terms of strategic complements, the outsider firms will also increase their product prices, but the magnitude of price increase will be less than that for cross-owners. As a result, each of the outsider firms tends to extract from its resource stock faster than each of the insider firms.

Using the parameter values $\rho = 0.1$ and $\gamma = 1$, Figure 5 plots the exhaustion dates $(T_i, T_o)$ as a function of the initial resource stock $S$, of a typical insider firm $i \in I$ that engages in cross-ownership, and an outsider firm $o \in O$ that remains independent, respectively, for $k = 8$, $n = 10$, and $v = 0.1$. Simulations using any combinations of $k, n$ with $0 < v < \frac{1}{k-1}$ and various values for the parameters $\rho$ and $\gamma \geq 0$ show that this result is qualitatively robust: a symmetric cross-ownership among a subset of firms will induce them to exhaust their stocks later than those of non-participants for any resource stock level.

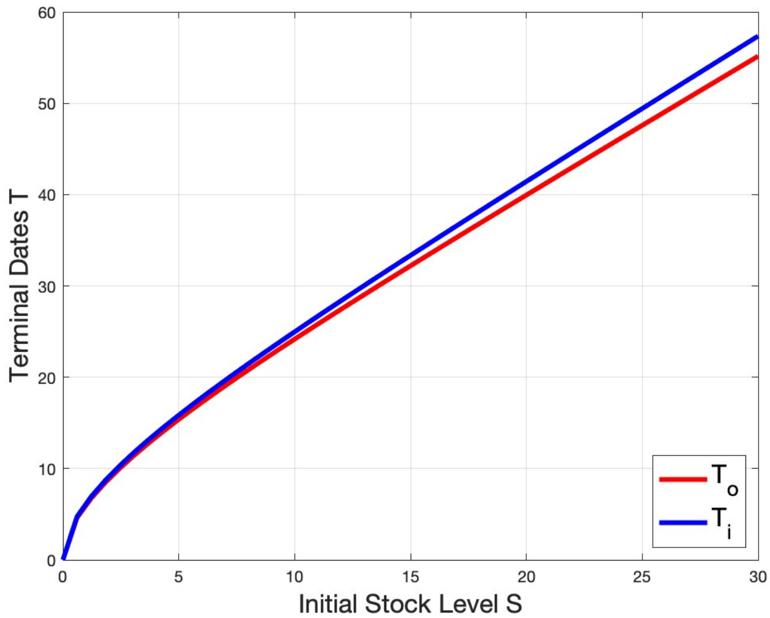
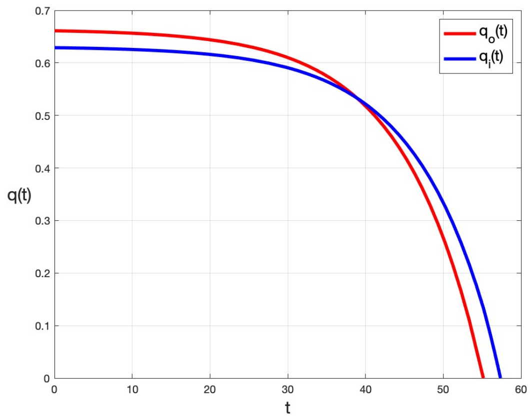
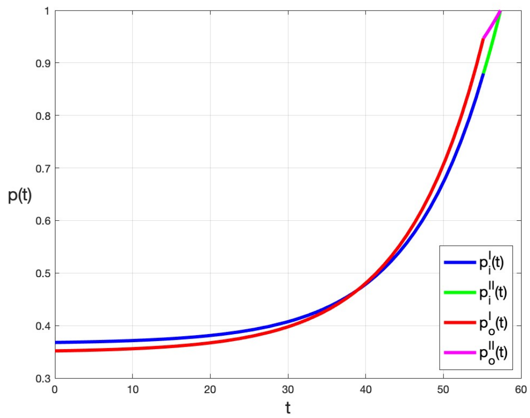



Figure 5: Terminal dates as a function of initial stock

Then the equilibrium price and extraction path consist of two phases: phase I from date 0 to $T_o$, and phase II from $T_o$ to $T_i$. During phase I, the extractions of all the $n$ firms are positive with prices increasing until $T_o$, where the extraction and the stock of firms $o \in O$ vanish. During phase II, only the remaining firms $i \in I$ still own a positive stock with prices rising further until $T_i$, where the extraction and the stock of these cross-owners vanish. Using the same parameter values as in Figure 5, we plot the equilibrium extraction path in Figure 6a and the equilibrium price path in Figure 6b, of a typical insider firm $i \in I$ and an outsider firm $o \in O$, respectively for $n = 10, k = 8, v = 0.1$ and $S = 30$. As shown in Figure 6, the outsiders start with a higher exploitation rate but lower product price than the cross-owners, but at some point, this trend reverses. As more resources get depleted, both outsiders and insiders gradually decrease their production and increase their product prices. Eventually,




(a) The OL-NBCOE extraction path




(b) The OL-NBCOE price path

Figure 6: The open-loop Nash-Bertrand cross-ownership equilibrium (OL-NBCOE)

when the outsiders deplete their resource stocks, the resource is supplied only by the group of cross-owners. As a result, the degree of concentration in supply increases over time. This increased concentration induced by cross-ownership confers market power on those cross-owners. Consequently, the prices are raised even further. With the remaining stocks, the insiders gradually decrease their production until the resource is totally depleted.

# 3 The profitability analysis

In this section, we exploit the characterization of both the static equilibrium in a differentiated Bertrand oligopoly and the OL-NBCOE in the above-defined dynamic game to investigate the profitability of the cross-ownership. We define the static profitability of cross-ownership as the difference between the equilibrium operating profits with and without cross-ownership, and the dynamic one as the difference between the equilibrium discounted sum of operating profits with and without cross-ownership.

## 3.1 The static case

The equilibrium operating profit for a typical firm $i$ that participates in cross-ownership is given by

$$\pi_i^v = p_i^v q_i^v = \frac{n\big(1-(k-2)v\big)\big((1-(k-2)v)(1+\gamma)n-(1+v)\gamma\big)\big(2n+(2n-1)\gamma\big)^2}{(A+Bv)^2},$$

while for a typical firm in the standard Bertrand model without cross-ownership is

$$\pi_b = p_b q_b = \frac{n\big(n+(n-1)\gamma\big)}{\big(2n+(n-1)\gamma\big)^2}.$$

A $k$-symmetric cross-ownership is profitable if

$$G(\gamma, k, n, v) = \pi_i^v - \pi_b > 0.$$

We derive the following result:

**Proposition 3.** *For any $\gamma \geq 0$, $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$, the static profitability of a $k$-symmetric cross-ownership for Bertrand competitors is always positive.*

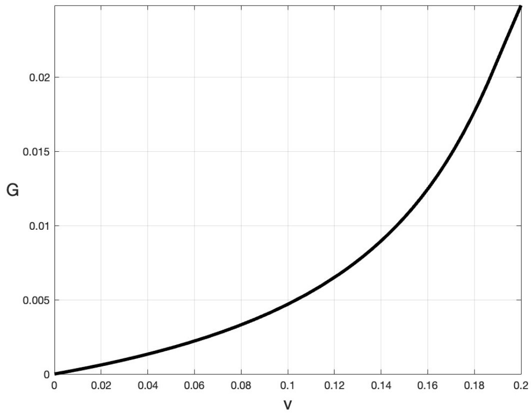
*Proof.* See Appendix C. □

Proposition 3 demonstrates that a $k$-symmetric cross-ownership is always profitable under Bertrand competition in a differentiated oligopolistic setting, irrespective of $k, n, v$ and $\gamma$. This is in sharp contrast with the case under Cournot competition, where the profitability of cross-ownership depends on three countervailing effects. One is the positive effect on cross-owners' profits due to the partial elimination of previous rivalry; the second is the negative effect of non-participants' production expansion in terms of strategic substitutability; and the last one is how aggressively outsiders will respond depending on the levels of shareholdings. For a cross-ownership to be profitable in Cournot competition, either the first effect dominates the latter two effects, or the first effect and third effect dominate the second one (Dai, Benchekroun and Long, 2022). However, in our Bertrand setting in which prices are strategic complements, the second effect becomes positive. Therefore, the three forces that drive the profitability of cross-ownership moves in the same direction, ensuring a positive profitability.

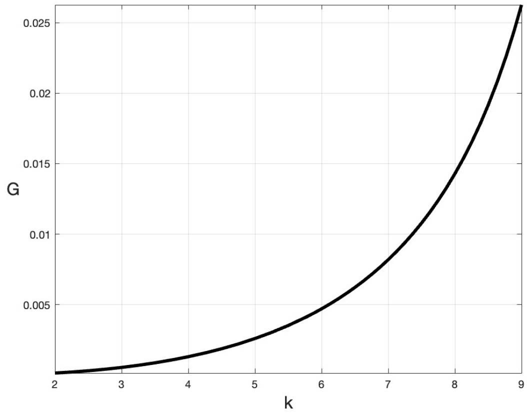
Furthermore, we plot the static profitability of cross-ownership as a function of $v$ when $\gamma = 3, k = 6$ and $n = 9$ in Figure 7a, of $k$ when $\gamma = 3, n = 9, v = 0.1$ in Figure 7b, of $n$ when $\gamma = 3, k = 6, v = 0.1$ in Figure 7c, and of $\gamma$ when $k = 6, n = 9, v = 0.1$ in Figure 7d, respectively. We observe:

**Result 2.** *Cross-ownership is increasingly profitable in either the levels $v$ or ratios $k/n$. However, the profitability of cross-ownership is non-monotone in the substitutability parameter $\gamma$: first increasing but later decreasing.*

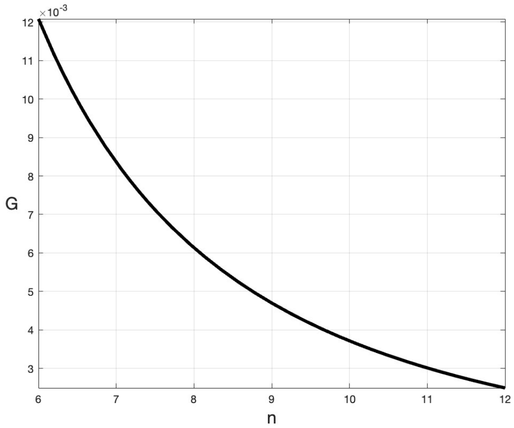
This result bears some similarity with Deneckere and Davidson (1985), which demonstrates that mergers among a differentiated oligopoly engaging in price competition are always beneficial and are increasing in size and that they are most beneficial when $\gamma$ takes on intermediate values. As in our case, an increase in either the ownership level $v$ or participation ratios $(k/n)$ leads to reduced competition and increased product prices for all firms, which undoubtedly increases the profitability of cross-ownership. Meanwhile, rival cross-shareholdings are most beneficial when $\gamma$ takes on intermediate values. Indeed, when $\gamma$ is close to 0, goods are basically unrelated, and each firm is charging the monopoly price for its products. Cross-ownership thus does not alter the degree of competition in the market and generates zero profitability. As products become slightly similar, competition pressure builds up, but cross-ownership



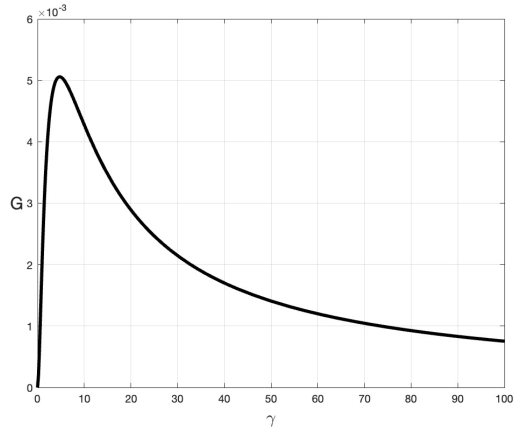

(a) $G$ as a function $v$




(b) $G$ as a function $k$




(c) $G$ as a function $n$




(d) $G$ as a function $\gamma$

Figure 7: The static profitability of cross-ownership under Bertrand competition

partially reduces that pressure, as one firm's lost profits can be recouped from its ownership stakes in its rival competitors. This decreased product differentiation gradually increases the profitability of cross-ownership until it reaches the maximum level. However, as product differentiation further decreases, the increased competition effect dominates the ownership-induced competition reduction effect, reducing the profitability of cross-ownership. Eventually, when $\gamma$ becomes very large, goods become perfect substitutes, and each firm is pricing at the marginal cost assumed at zero. Therefore, cross-ownership does not significantly reduce the degree of competition in the market and generates no benefits. Simulations using any other combinations of $k, n, v, \gamma$ confirm that Result 2 is qualitatively robust.

## 3.2 The dynamic case

We now turn to the profitability analysis of cross-ownership in a nonrenewable resource industry. The equilibrium discounted sum of operating profits for a typical firm $i$ that engages in rival cross-shareholdings is given by:

$$V_S = \int_0^{T_o} e^{-\rho t}\left[p_i^I(t)q_i^I(t)\right]dt + \int_{T_o}^{T_i} e^{-\rho t}\left[p_i^{II}(t)q_i^{II}(t)\right]dt,$$

where the phase I and II equilibrium price paths $(p_i(t), p_o(t))$ are given by (6) and (7), the extraction paths are given by (8) and (9), and the exhaustion dates $(T_i, T_o)$ are solutions to (10) and (11), respectively. The equilibrium discounted sum of profits without cross-ownership for an individual firm is given by:

$$V_B = \int_0^{T_B} e^{-\rho t}\left[p_B(t)q_B(t)\right]dt,$$

where

$$p_B(t) = \frac{n + (n + (n-1)\gamma)e^{\rho(t-T_B)}}{2n + (n-1)\gamma}, \tag{12}$$

$$q_B(t) = \frac{n + (n-1)\gamma}{2n + (n-1)\gamma}\left[1 - e^{\rho(t-T_B)}\right], \tag{13}$$

and $T_B$ is the solution to

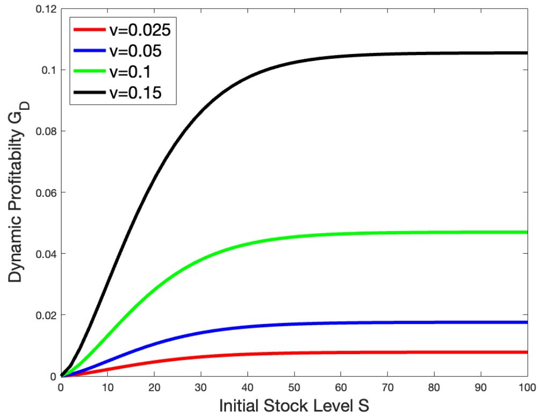
$$\int_0^{T_B} q_B(t)dt = \frac{n + (n-1)\gamma}{2n + (n-1)\gamma}\left(T_B - \frac{1}{\rho} + \frac{e^{-\rho T_B}}{\rho}\right) = S. \tag{14}$$

Then a $k$-symmetric cross-ownership is profitable when

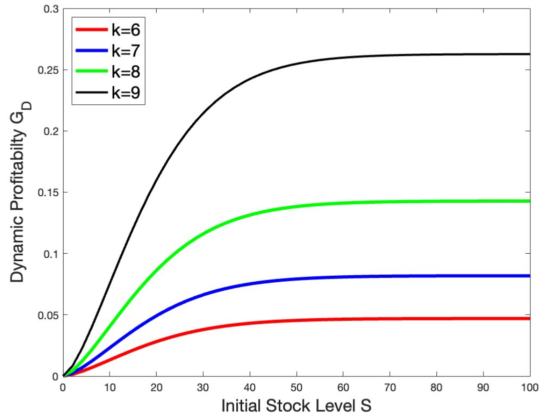
$$G_D(\gamma, k, n, v, S) = V_S - V_B > 0.$$

It will be useful to explicitly express $G_D$ as a function of $(\gamma, k, n, v, S)$, but its expression is too cumbersome to report here. Instead, we choose to numerically examine the dynamic profitability of the $k$-symmetric cross-ownership. Using the same parameter value $\rho = 0.1$, we illustrate the gains resulting from a $k$-symmetric cross-ownership as a function of initial stock $S$, for different levels of $v$ when $\gamma = 3$, $k = 6$ and $n = 9$ in Figure 8a, for different numbers of $k$ when $\gamma = 3$, $v = 0.1$ and $n = 9$ in Figure 8b, for different numbers of $n$ when $\gamma = 3$, $k = 6$ and $v = 0.1$ in Figure 8c, and for different levels of $\gamma$ when $v = 0.1$, $k = 6$ and $n = 9$ in Figure 8d, respectively. Figure 8 shows that it is always profitable for firms to participate in cross-ownership for any levels of initial resource stock $S$, irrespective of $\gamma, k, n$ and $v$. In addition, for any initial stock $S$, the higher the level of cross-ownership $v$ or participation ratios $k/n$, the higher the dynamic profitability. However, the profitability of cross-ownership first increases but then decreases in the $\gamma$ for all $S$. These findings are consistent with the ones obtained under the static case.

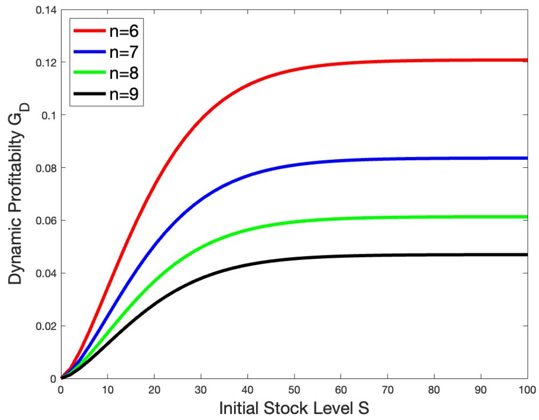
To make a direct comparison between the static and dynamic profitability, we use the same set of parameter values as in Figure 8 and plot in Figure 9 the static and



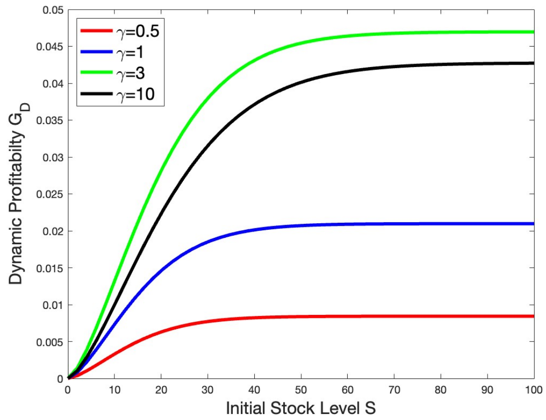

(a) $G_D$ as a function of $S$ when $v$ varies




(b) $G_D$ as a function of $S$ when $k$ varies




(c) $G_D$ as a function of $S$ when $n$ varies




(d) $G_D$ as a function of $S$ when $\gamma$ varies

Figure 8: The dynamic profitability of cross-ownership under Bertrand competition

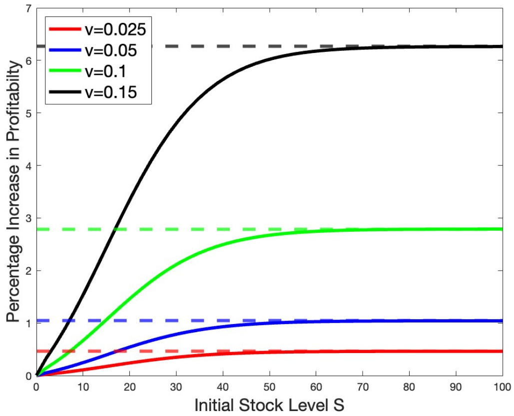
dynamic percentage profit gains resulting from a $k$-symmetric cross-ownership, which are respectively defined by

$$d(\gamma, k, n, v) = \frac{\pi_i^v - \pi_b}{\pi_b}, \quad D(\gamma, k, n, v, S) = \frac{V_S - V_B}{V_B}.$$

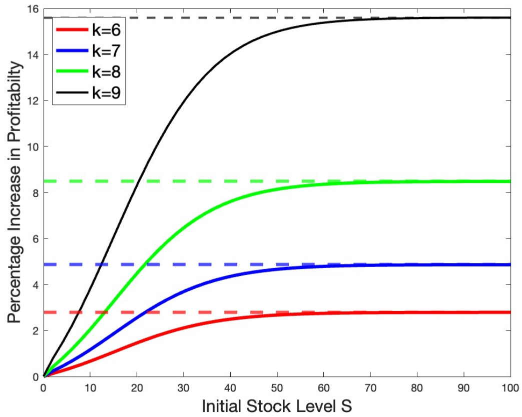
The dashed and solid lines denote the percentage profit gains in the static and dynamic cases, respectively. It can be easily observed that when $S$ is large enough, the dynamic percentage profit gain asymptotically converges to the static result. However, when $S$ is small enough, cross-ownership may turn out to be less profitable in a nonrenewable resource sector than in other sectors without stock constraints. Simulations using a wide range of values of $\gamma, k$ and $n$ with $v < \frac{1}{k-1}$ and of the parameter $\rho$ show that this result is qualitatively robust. We can thus summarize this result in the following:

**Result 3.** *When the initial resource stock owned by each firm is small enough, a k-symmetric cross-ownership yields less profit gains in the case of a differentiated nonrenewable resource oligopoly than other industries without stock constraints.*

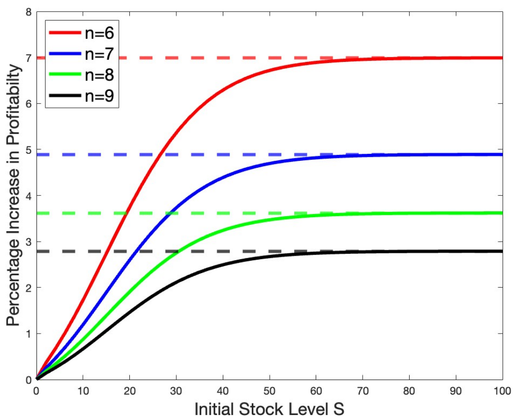
Result 3 sharply contrasts with the findings obtained in the resource game under Cournot competition, where the presence of resource scarcity makes it more likely to be



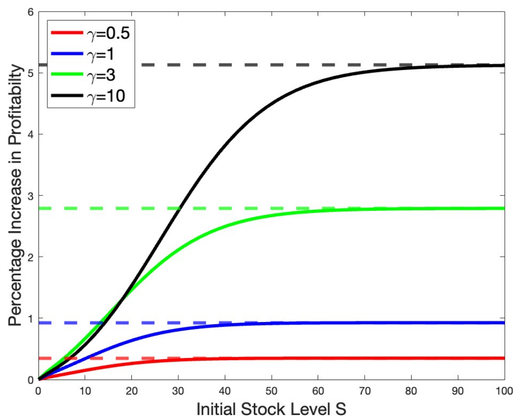

(a) Profitability as a function $v$




(b) Profitability as a function $k$




(c) Profitability as a function $n$




(d) Profitability as a function $\gamma$

Figure 9: The percentage increase in profitability of cross-ownership

profitable even under the scenario for which it is strictly unprofitable in the corresponding static framework (Benchekroun, Breton and Chaudhuri, 2019; Dai, Benchekroun and Long, 2022). Unlike in the quantity competition setting where outsiders are more constrained in their response to any output reduction brought by insiders due to their limited resource stocks, here under price competition, resource constraints tend to work in the opposite direction by inducing the outsiders to aggressively lower their product price and expand their production. Consequently, this intensified competition reduces the profitability of cross-ownership in a differentiated nonrenewable resource oligopoly when the resource stock is small.

# 4 Welfare analysis

In this section, we investigate the impact of cross-ownership on social welfare, which is defined as the sum of consumer surplus (CS) and producer surplus (PS) or industry profits. The latter is defined as the combined sum of the operating profits of the cross-owners, $k\pi_i^v$, and of the outsiders, $(n-k)\pi_o^v$. We first look at the static case, and then we move to the case of a differentiated nonrenewable resource industry. Subsequently, we make a comparison of the welfare changes between the dynamic and static cases.

Note that the CS is defined by

$$CS = \sum_{j=1}^{n} q_j - \frac{n+\gamma}{2n(1+\gamma)} \sum_{j=1}^{n} q_j^2 - \frac{\gamma}{n(1+\gamma)} \sum_{j \neq i} q_i q_j - \sum_{j=i}^{n} p_j q_j.$$

Using symmetry, the CS under the $k$-symmetric cross-ownership structure is given by

$$\begin{aligned} CS_v = & \left(kq_i^v + (n-k)q_o^v\right) - \frac{n+\gamma}{2n(1+\gamma)} \left(k(q_i^v)^2 + (n-k)(q_o^v)^2\right) - \left(kp_i^v q_i^v + (n-k)p_o^v q_o^v\right) \\ & - \frac{\gamma}{n(1+\gamma)} \left(\frac{k(k-1)}{2}(q_i^v)^2 + k(n-k)q_i^v q_o^v + \frac{(n-k)(n-k-1)}{2}(q_o^v)^2\right), \end{aligned}$$

and the CS without cross-ownership is

$$CS_b = \frac{n}{2} q_b^2 = \frac{n}{2} \left(\frac{n+(n-1)\gamma}{2n+(n-1)\gamma}\right)^2,$$

while the PS with and without cross-ownership are respectively given by

$$\begin{aligned} PS_v &= k\pi_i^v + (n-k)\pi_o^v = kp_i^v q_i^v + (n-k)p_o^v q_o^v, \\ PS_b &= n\pi_b = np_b q_b. \end{aligned}$$

Therefore, the static welfare change induced by the $k$-symmetric cross-ownership can be expressed as

$$\Delta W(\gamma, k, n, v) = W_v - W_b = CS_v + PS_v - (CS_b + PS_b),$$

with the percentage welfare change defined by

$$W(\gamma, k, n, v) = \frac{W_v - W_b}{W_b}.$$

**Proposition 4.** *For any $\gamma \geq 0$, $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$, a $k$-symmetric cross-ownership in a differentiated Bertrand oligopoly is never welfare-improving.*

*Proof.* See Appendix D. □

When firms participate in these cross-ownership arrangements, they tend to compete less aggressively with each other and thus unilaterally increase their product prices and reduce their outputs, since any profit gains from the firms' own activities may be offset by a negative impact on the target firms' profits. Although the outsiders raise their product prices and expand their outputs as a response, the overall reduction in the CS dominates any profit gains brought by cross-ownership, thereby resulting in a welfare loss for society.

We now conduct the welfare analysis for a nonrenewable industry. Under cross-ownership, the total surplus generated by the exploitation of a nonrenewable resource is given by

$$W_S = \int_0^{T_o} e^{-\rho t} \Big[ CS_S^I(t) + PS_S^I(t) \Big] dt + \int_{T_o}^{T_i} e^{-\rho t} \Big[ CS_S^{II}(t) + PS_S^{II}(t) \Big] dt,$$

where

$$\begin{aligned} CS_S(t) =& kq_i(t) + (n-k)q_o(t) - \frac{n+\gamma}{2n(1+\gamma)} \Big( k(q_i(t))^2 + (n-k)(q_o(t))^2 \Big) - kp_i(t)q_i(t) - (n-k)p_o(t)q_o(t) \\ & - \frac{\gamma}{n(1+\gamma)} \left( \frac{k(k-1)(q_i(t))^2}{2} + k(n-k)q_i(t)q_o(t) + \frac{(n-k)(n-k-1)(q_o(t))^2}{2} \right), \\ PS_S(t) =& kp_i(t)q_i(t) + (n-k)p_o(t)q_o(t), \end{aligned}$$

and the phase I and II equilibrium price paths $(p_i(t), p_o(t))$ are given by (6) and (7), the extraction paths are given by (8) and (9), and the exhaustion dates $(T_i, T_o)$ are solutions to (10) and (11), respectively. The social welfare generated by the exploitation of the nonrenewable resource under the standard Bertrand model without cross-ownership is given by

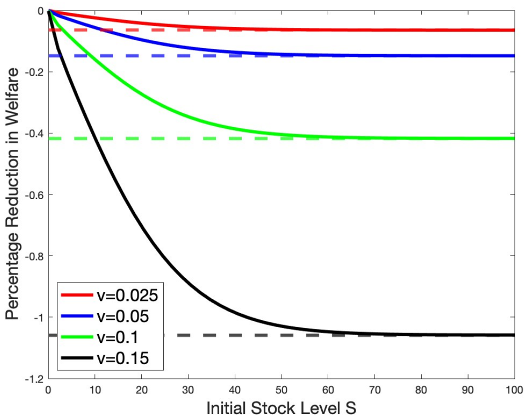
$$W_B = \int_0^{T_B} e^{-\rho t} \left[ \frac{n}{2}(q_B(t))^2 + np_B(t)q_B(t) \right] dt,$$

where $p_B(t), q_B(t)$ and $T_B$ are defined by (12), (13) and (14), respectively. The percentage welfare change in a differentiated nonrenewable resource industry can thus be defined as

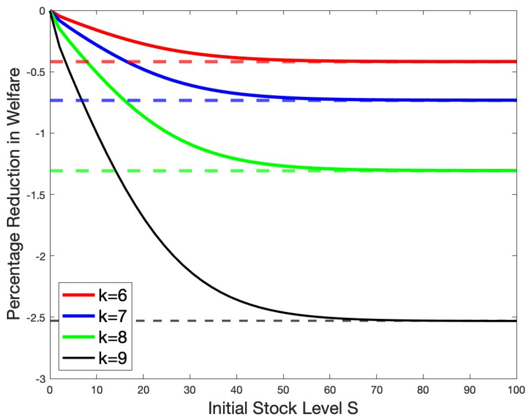
$$W_D = \frac{W_S - W_B}{W_B}.$$

Next, we numerically examine the percentage welfare change of the $k$-symmetric cross-ownership in the dynamic case and compare it with the static one. Using the same parameter value $\rho = 0.1$, we illustrate the dynamic percentage welfare change resulting from a $k$-symmetric cross-ownership as a function of initial stock $S$, for different levels of $v$ when $\gamma = 3$, $k = 6$ and $n = 9$ in Figure 10a, for different numbers of $k$ when $\gamma = 3$, $v = 0.1$ and $n = 9$ in Figure 10b, for different numbers of $n$ when $\gamma = 3$, $k = 6$ and $v = 0.1$ in Figure 10c, and for different levels of $\gamma$ when $v = 0.1$, $k = 6$ and $n = 9$ in Figure 10d, respectively. The dashed and solid lines in Figure 10 denote the percentage welfare loss in the static and dynamic cases, respectively.

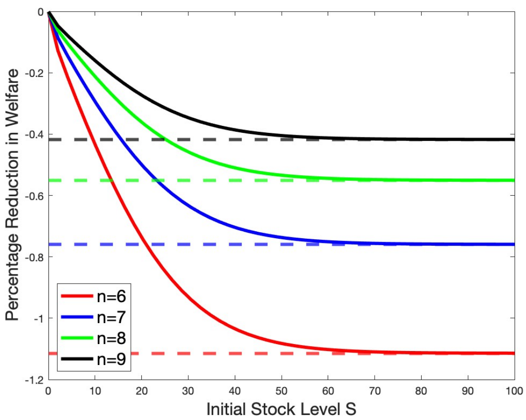
It can be easily observed that when $S$ is large enough, i.e., the resource is abundant, the dynamic percentage welfare change asymptotically converges to the static result. However, when $S$ is small enough, the dynamic percentage welfare loss in the dynamic case turns out to be smaller than that of the static case. Simulations using a wide range of values of $\gamma, k$ and $n$ with $v < \frac{1}{k-1}$ and of the parameter $\rho$ show that this result is qualitatively robust.



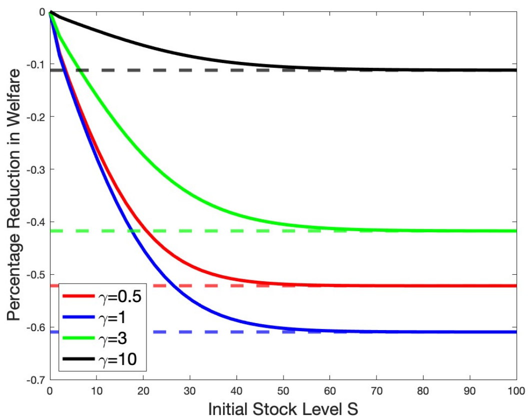

(a) $W_D$ as a function of $S$ when $v$ varies




(b) $W_D$ as a function of $S$ when $k$ varies




(c) $W_D$ as a function of $S$ when $n$ varies




(d) $W_D$ as a function of $S$ when $\gamma$ varies

Figure 10: The percentage welfare change as a function of initial stock

**Result 4.** *When the initial resource stock owned by each firm is small enough, the percentage welfare loss resulting from a k-symmetric cross-ownership in the case of a differentiated nonrenewable resource oligopoly is smaller than that of other industries without stock constraints.*

This result seems quite counterintuitive, as one would typically expect the opposite: the welfare loss should be larger in the dynamic case. This is because when the resource stock owned by each firm is small enough, cross-shareholdings between rival firms will induce them to slow down their extraction and eventually monopolize the market after the outsiders have exhausted their resource stocks. As a result, the concentrated supply will enable cross-owners to charge much higher prices, leading to greater welfare losses.

While cross-ownership participants can raise their product prices much higher, it also prolongs the duration over which the resources can be exploited. As resources become increasingly scarce, the availability and extended periods of use of these resources partially offset the negative effect of higher prices on the consumer surplus. Thus, the consumer surplus loss is relatively smaller in the dynamic case than in the static framework when $S$ is small enough. Consequently, the smaller loss in consumer surplus due to increased scarcity and the increased profits due to higher prices will result in a smaller welfare loss in the case of a differentiated nonrenewable resource oligopoly than that in the static case.

# 5 Conclusion

In this paper, we examine whether the conclusions obtained in a static framework can be extended to the case of a differentiated nonrenewable resource oligopoly when rival firms compete *à* la Bertrand and engage in cross-shareholdings. We show that in the static setting, a symmetric cross-ownership is always profitable and is increasingly profitable in either levels or ratios, and it is most beneficial when the degree of competition takes on intermediate values. These static results also carry over to the case of a dynamic nonrenewable differentiated oligopoly. However, when the initial resource stock owned by each firm is small enough, a $k$-symmetric cross-ownership yields less profit gains in the case of a differentiated nonrenewable resource oligopoly than other industries without stock constraints. This is in sharp contrast with the games in quantity setting. Finally, we demonstrate that cross-ownership may turn out to be relatively less detrimental to society in a differentiated nonrenewable resource industry than other industries where resource constraints are absent.

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# Appendices

## A Proof of Proposition 1

*Proof.* Note that $F(v) = A + Bv$ is a linear function in $v \in (0, \frac{1}{k-1})$ with $F(0) = A > 0$ and

$$F(\frac{1}{k-1}) = \frac{(2n+k-2)(n-k)\gamma^2 + 2n(3n-k-1)\gamma + 4n^2}{k-1} > 0.$$

Therefore, for any $\gamma \geq 0$, $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$, $F(v) = A + Bv > 0$. Moreover, we have

$$p_i^v - p_o^v = \frac{nv\gamma(k-1)}{A+Bv} > 0, \quad q_i^v - q_o^v = \frac{nv\gamma(k-1)(1+\gamma)}{A+Bv} > 0,$$

and

$$\begin{aligned}
\frac{\partial p_i^v(v)}{\partial v} &= \frac{n\gamma(k-1)\big(2n+(2n-1)\gamma\big)\big(2n+(n+k-1)\gamma\big)}{(A+Bv)^2} > 0,\\
\frac{\partial p_o^v(v)}{\partial v} &= \frac{kn\gamma^2(k-1)\big(2n+(2n-1)\gamma\big)}{(A+Bv)^2} > 0,\\
\frac{\partial q_i^v(v)}{\partial v} &= -\frac{\gamma(k-1)\big(2n+(2n-1)\gamma\big)\big((n-k)(n-1)\gamma^2+(3n-k-1)n\gamma+2n^2\big)}{(A+Bv)^2} < 0,\\
\frac{\partial q_o^v(v)}{\partial v} &= \frac{k\gamma^2(k-1)\big(n+(n-1)\gamma\big)\big(2n+(2n-1)\gamma\big)}{(A+Bv)^2} > 0.
\end{aligned}$$

Thus, for any $v \in (0, \frac{1}{k-1})$,

$$p_i^v(v) > p_o^v(v) > p_o^v(0) = p_b, \quad q_o^v(v) > q_o^v(0) = q_b = q_i^v(0) > q_i^v(v),$$

and $p_i^v$, $p_o^v$ and $q_o^v$ are strictly increasing in $v$, but $q_i^v$ decreases in $v$. $\square$

## B Proof of Proposition 2

*Proof.* We use optimal control theory to characterize the OL-NBCOE. The current value Hamiltonian associated with the problem of a typical firm $i \in I$ is given by

$$H_i(p_i, p_{-i}, \lambda_i, t) = \frac{1}{f(v)}\left(\big(1-(k-2)v\big)p_i q_i + v\sum_{m\in I\setminus i} p_m q_m\right) - \lambda_i q_i,$$

while that for a typical firm $o \in O$ is

$$H_o(p_o, p_{-o}, \lambda_o, t) = p_o q_o - \lambda_o q_o.$$

Exploiting symmetry, the maximum principle yields

$$\Big(1-(k-2)v\Big)\Big[1-2p_i(1+\gamma)+\frac{\gamma}{n}(kp_i+(n-k)p_o)+p_i\frac{\gamma}{n}\Big]+v(k-1)p_i\frac{\gamma}{n}$$
$$+\lambda_i\Big(1-(k-2)v-(k-1)v^2\Big)\Big(1+\gamma(1-\frac{1}{n})\Big)=0, \quad (15)$$

for $i=1,2,\cdots,k$ and

$$1-2p_o(1+\gamma)+\frac{\gamma}{n}\Big(kp_i+(n-k)p_o\Big)+p_o\frac{\gamma}{n}+\lambda_0\Big(1+\gamma(1-\frac{1}{n})\Big)=0, \quad (16)$$

for $o=k+1,\cdots,n$, with

$$\frac{d\lambda_i}{dt}=\rho\lambda_i, \quad (17)$$
$$\frac{d\lambda_o}{dt}=\rho\lambda_o. \quad (18)$$

Solving for $(p_i,p_o)$ from (15) and (16), then we get

$$p_i(t)=\frac{n\big(1-(k-2)v\big)\big(2n(1+\gamma)-\gamma\big)+X\lambda_i+Y\lambda_o}{A+Bv}, \quad (19)$$

where

$$X=\big(1-(k-2)v-(k-1)v^2\big)\big(n(1+\gamma)-\gamma\big)\big((n+k-1)\gamma+2n\big),$$
$$Y=\big(1-(k-2)v\big)\big(n(1+\gamma)-\gamma\big)(n-k)\gamma,$$

and

$$p_o(t)=\frac{n\Big((1-(k-2)v)2n(1+\gamma)-(1+v)\gamma\Big)+Z\lambda_i+\Gamma\lambda_o}{A+Bv}, \quad (20)$$

where

$$Z=\big(1-(k-2)v-(k-1)v^2\big)\big(n(1+\gamma)-\gamma\big)k\gamma,$$
$$\Gamma=\Big(\big(1-(k-2)v\big)\big(2n(1+\gamma)-k\gamma\big)-(1+v)\gamma\Big)\big(n(1+\gamma)-\gamma\big).$$

During the second phase where only firms $i\in I$ extract a positive quantity, the maximum principle yields

$$\Big(1-(k-2)v\Big)\Big[1-2p_i(1+\gamma)+\frac{\gamma}{n}(kp_i+(n-k)p_o)+p_i\frac{\gamma}{n}\Big]+v(k-1)p_i\frac{\gamma}{n}$$
$$+\lambda_i\Big(1-(k-2)v-(k-1)v^2\Big)\Big(1+\gamma(1-\frac{1}{n})\Big)=0, \quad (21)$$

with

$$\frac{d\lambda_i}{dt}=\rho\lambda_i, \quad (22)$$

where $p_o$ is the solution to

$$q_o = 1 - p_o(1+\gamma) + \frac{\gamma}{n}\Big(kp_i + (n-k)p_o\Big) = 0,$$

or

$$p_o = \frac{1 + \frac{\gamma}{n}kp_i}{1+\frac{\gamma}{n}k} = \frac{n + k\gamma p_i}{n + k\gamma}. \tag{23}$$

Substitute (23) into (21) and solve for $p_i$, we obtain

$$p_i(t) = \frac{\Big(1-(k-2)v\Big)(1+\gamma)n^2 + \Big(1-(k-2)v-(k-1)v^2\Big)\Big(n(1+\gamma)-\gamma\Big)(n+k\gamma)\lambda_i}{\Big(1-(k-2)v\Big)(1+\gamma)(2n^2+k\gamma n) - (1+v)(n\gamma + k\gamma^2)}. \tag{24}$$

The terminal dates $T_i$ and $T_o$ are endogenous and determined by

$$H_i(p_i(T_i), p_{-i}(T_i), \lambda_i(T_i), T_i) = 0$$

for $i \in I$ and

$$H_o(p_o(T_o), p_{-o}(T_o), \lambda_o(T_o), T_o) = 0$$

for $o \in O$. These terminal conditions, along with the maximum principle, imply that

$$q_i(T_i) = 0, \quad q_o(T_o) = 0. \tag{25}$$

From (17), (18) and (22) and continuity of the costate variable $\lambda_i$ at $T_o$, we have

$$\lambda_i = \lambda_{i0}e^{\rho t} \quad \forall\, t \in [0, T_i], \tag{26}$$

$$\lambda_o = \lambda_{o0}e^{\rho t} \quad \forall\, t \in [0, T_o], \tag{27}$$

where $\lambda_{i0}$ and $\lambda_{o0}$ are determined using conditions (25) along with (24) and (20).

Note that at the terminal date $T_i$,

$$q_i(T_i) = 1 - p_i(1+\gamma) + \frac{\gamma}{n}\Big(kp_i + (n-k)p_o\Big) = \frac{(1+\gamma)n}{n+k\gamma}(1 - p_i(T_i)) = 0,$$

or

$$p_i(T_i) = 1.$$

Therefore, from (24), we have

$$p_i(T_i) = \frac{\Big(1-(k-2)v\Big)(1+\gamma)n^2 + \Big(1-(k-2)v-(k-1)v^2\Big)\Big(n(1+\gamma)-\gamma\Big)(n+k\gamma)\lambda_{i0}e^{\rho T_i}}{\Big(1-(k-2)v\Big)(1+\gamma)(2n^2+k\gamma n) - (1+v)(n\gamma + k\gamma^2)} = 1. \tag{28}$$

That is,

$$\lambda_{i0} = \frac{\Big[\big(1-(k-2)v\big)(1+\gamma)n - (1+v)\gamma\Big]}{\big(1-(k-2)v-(k-1)v^2\big)\big(n(1+\gamma)-\gamma\big)} e^{-\rho T_i},$$

and thus

$$\lambda_i = \lambda_{i0}e^{\rho t} = \frac{\Big[\big(1-(k-2)v\big)(1+\gamma)n - (1+v)\gamma\Big]}{\big(1-(k-2)v-(k-1)v^2\big)\big(n(1+\gamma)-\gamma\big)} e^{\rho(t-T_i)}. \tag{29}$$

Also, at the terminal date $T_o$, we have

$$q_o(T_o) = 1-(1+\gamma)p_o(T_o) + \frac{\gamma}{n}\Big(kp_i(T_o)+(n-k)p_o(T_o)\Big) = 0,$$

or

$$n + k\gamma p_i(T_o) - (n+k\gamma)p_o(T_o) = 0. \tag{30}$$

From (19) and (20),

$$p_i(T_o) = \frac{n\Big(1-(k-2)v\Big)\Big(2n(1+\gamma)-\gamma\Big) + X\lambda_i(T_o) + Y\lambda_{o0}e^{\rho T_o}}{A+Bv}, \tag{31}$$

$$p_o(T_o) = \frac{n\Big(\big(1-(k-2)v\big)2n(1+\gamma)-(1+v)\gamma\Big) + Z\lambda_i(T_o) + \Gamma\lambda_{o0}e^{\rho T_o}}{A+Bv}, \tag{32}$$

where

$$\lambda_i(T_o) = \frac{\Big[\big(1-(k-2)v\big)(1+\gamma)n - (1+v)\gamma\Big]}{\big(1-(k-2)v-(k-1)v^2\big)\big(n(1+\gamma)-\gamma\big)} e^{\rho(T_o-T_i)}.$$

Substitute (31) and (32) into (30), we can then solve for

$$\lambda_{o0} = \frac{k\gamma\Big[(1+v)\gamma - \big(1-(k-2)v\big)(1+\gamma)n\Big]e^{-\rho T_i} + n\Big[(1+v)\gamma - 2n\big(1-(k-2)v\big)(1+\gamma)\Big]e^{-\rho T_o}}{\Big[(1+v)(n+k\gamma)\gamma - \big(1-(k-2)v\big)(1+\gamma)(2n+k\gamma)n\Big]},$$

and thus

$$\lambda_o = \lambda_{o0}e^{\rho t} = \frac{k\gamma\Big[(1+v)\gamma - \big(1-(k-2)v\big)(1+\gamma)n\Big]e^{\rho(t-T_i)} + n\Big[(1+v)\gamma - 2n\big(1-(k-2)v\big)(1+\gamma)\Big]e^{\rho(t-T_o)}}{\Big[(1+v)(n+k\gamma)\gamma - \big(1-(k-2)v\big)(1+\gamma)(2n+k\gamma)n\Big]}. \tag{33}$$

Substituting (29) and (33) into (19), (20) and (24) yields the Phase I ($0 \le t \le T_o$) and Phase II ($T_o \le t \le T_i$) equilibrium price paths of all the firms as presented in (6) and (7). These equilibrium paths are determined as functions of the terminal dates $T_i$ and $T_o$, which are then endogenously determined from the resource constraint conditions as in (10) and (11). It can be shown that such a non-linear system in $(T_i, T_o)$ admits a unique solution with $T_i \geq T_o$. $\square$

# C Proof of Proposition 3

*Proof.* We have

$$G(\gamma, k, n, v) = \pi_i^v - \pi_b = \frac{\gamma^2 nv(k-1)\Theta(v)}{\big(2n+(n-1)\gamma\big)^2(A+Bv)^2},$$

where

$$\Theta(v) = \theta_0 - \Big[(n-1)\gamma^3\theta_1 + n\gamma^2\theta_2 + 4\gamma n^2\theta_3 + 4n^3(k-1)(2k-3)\Big]v,$$

with

$$\begin{aligned}
\theta_0 &= \big(2n+(2n-1)\gamma\big)\big(2n+(n-1)\gamma\big)\big(2n(k-1)+(n-1)(2k-1)\gamma\big) > 0,\\
\theta_1 &= (k^2+1)(2n-1)^2 + k^2(k-2) - (9k+1)n^2 + k(11n-2),\\
\theta_2 &= k^2\big(4n(4n-5)+k+3\big) + (n-1)\big((15n-7)-k(37n-15)\big),\\
\theta_3 &= k^2(5n-3) - 2(3n-2)(2k-1).
\end{aligned}$$

Therefore, $G(\gamma, k, n, v)$ has the same sign as $\Theta(v)$. Note that $\Theta(v)$ is a linear function in $v$ with

$$\Theta(0) = \theta_0 > 0, \quad \Theta(\frac{1}{k-1}) = \frac{1}{k-1}\Omega(k, n, \gamma),$$

where

$$\begin{aligned}
\Omega(k,n,\gamma) =& n^3(1+\gamma)(2+\gamma)\big(2(k-1)+\gamma(3k-1)\big) - n^2\gamma\big(2(1+\gamma)(2+\gamma)(k^2-1)+k(5\gamma^2+10\gamma+4)\big)\\
&- n\gamma^2\big((k^3+1)(1+\gamma) - k^2(7+5\gamma) + k(1-\gamma)\big) + k(k^2-3k+1)\gamma^3.
\end{aligned}$$

At $n = k$, we have

$$\Omega(k, k, \gamma) = k(k-1)(2k+(2k-1)\gamma)^2 > 0.$$

In addition, the derivative of $\Omega(k, n, \gamma)$ with respect to $n$ is given by

$$\begin{aligned}
\frac{\partial\Omega(k,n,\gamma)}{\partial n} =& 3n^2(1+\gamma)(2+\gamma)\big(2(k-1)+\gamma(3k-1)\big) - 2n\gamma\big(2(1+\gamma)(2+\gamma)(k^2-1)+k(5\gamma^2+10\gamma+4)\big)\\
&- \gamma^2\big((k^3+1)(1+\gamma) - k^2(7+5\gamma) + k(1-\gamma)\big),
\end{aligned}$$

which is a quadratic U-shaped function of $n$ with

$$\left.\frac{\partial \Omega(k,n,\gamma)}{\partial n}\right|_{n=k} = (2k+(2k-1)\gamma)\big((2k-1)(k-1)\gamma^2+(8k^2-7k+1)\gamma+6k(k-1)\big) > 0.$$

Therefore, for any $\gamma \geq 0$ and $n \geq k \geq 2$, $\frac{\partial \Omega(k,n,\gamma)}{\partial n} > 0$. This means that $\Omega(k,n,\gamma)$ is a strictly increasing function in $n$ with $\Omega(k,k,\gamma) > 0$. Then, we must have $\Omega(k,n,\gamma) > 0$ and thus $\Theta(\frac{1}{k-1}) > 0$ for all $\gamma \geq 0$ and $n \geq k \geq 2$. This condition, together with the fact that $\Theta(0) = \theta_0 > 0$ and $\Theta(v)$ is a linear function in $v$, concludes that $\Theta(v) > 0$ and thus $G(\gamma,k,n,v)$ for all $\gamma \geq 0$, $n \geq k \geq 2$, and $v \in (0, \frac{1}{k-1})$. □

# D Proof of Proposition 4

*Proof.* After substitution, the static welfare change becomes

$$\Delta W = \frac{k(k-1)nv\gamma\Phi(v)}{2(2n+(n-1)\gamma)^2\Big(2n^2(1-(k-2)v)(\gamma^2+3\gamma+2)-n\gamma(4+3\gamma+(6+5\gamma-2k(1+\gamma))v)+(1-(k^2-k-1)v)\gamma^2\Big)^2},$$

where

$$\Phi(v) = \phi_0 + \Big[16(k-2)n^4 - (n-k)(k-1)(n-1)^2\gamma^4 + n\gamma^3\phi_1 + 4n^2\gamma^2\phi_2 + 4n^3\gamma\phi_3\Big]v,$$

with

$$\begin{aligned}
\phi_0 &= -2n\big(2n+(n-1)\gamma\big)\big(2n+(2n-1)\gamma\big)^2 < 0,\\
\phi_1 &= (5n-4)k^2 + (8n^3-17n^2+5n+3)k - (16n^3-33n^2+20n-3),\\
\phi_2 &= 8(k-2)n^2 + (19-9k)n + k^2 + k - 5,\\
\phi_3 &= 10(k-2)n + 11 - 5k.
\end{aligned}$$

Thus, $\Delta W(k,n,v,\gamma)$ has the same sign as $\Phi(v)$, which is a linear function in $v$ with

$$\Phi(0) = \phi_0 < 0, \quad \Phi(\frac{1}{k-1}) = \frac{1}{k-1}\Psi(k,n,\gamma),$$

where

$$\begin{aligned}
\Psi(k,n,\gamma) =& (k-1)k\gamma^4 - 8(1+\gamma)^2(2+\gamma)n^4 - \gamma(1+\gamma)\Big((k-1)\gamma^2 - 16\gamma - 4(k+5)\Big)n^3\\
&+\gamma^2\Big((k^2-2)(1+\gamma)(4+\gamma)+k(\gamma^2-5\gamma-8)\Big)n^2 + \gamma^3\Big(-2(2+\gamma)k^2+(5+\gamma)k+1+\gamma\Big)n,
\end{aligned}$$

At $n = k$,

$$\Psi(k,k,\gamma) = -k(4k+(k-1)\gamma)(2k+(2k-1)\gamma)^2 < 0.$$

In addition, we have

$$\frac{\partial \Psi(k,n,\gamma)}{\partial n} = -32(1+\gamma)^2(2+\gamma)n^3 - 3\gamma(1+\gamma)\Big((k-1)\gamma^2 - 16\gamma - 4(k+5)\Big)n^2 + 2\gamma^2\Big((k^2-2)(1+\gamma)(4+\gamma) + k(\gamma^2-5\gamma-8)\Big)n + \gamma^3\Big(-2(2+\gamma)k^2 + (5+\gamma)k + 1 + \gamma\Big),$$

and

$$\frac{\partial^2 \Psi(k,n,\gamma)}{\partial n^2} = -96(1+\gamma)^2(2+\gamma)n^2 - 6\gamma(1+\gamma)\Big((k-1)\gamma^2 - 16\gamma - 4(k+5)\Big)n + 2\gamma^2\Big((k^2-2)(1+\gamma)(4+\gamma) + k(\gamma^2-5\gamma-8)\Big),$$

which is an inverted U-shaped quadratic function in $n$ with

$$\left.\frac{\partial^2 \Psi(k,n,\gamma)}{\partial n^2}\right|_{n=k} = -4F(k,\gamma),$$

where

$$F(k,\gamma) = k^2\big(48+114\gamma+88\gamma^2+23\gamma^3+\gamma^4\big) - k\gamma\big(30+50\gamma+23\gamma^2+2\gamma^3\big) + \gamma^2\big(4+5\gamma+\gamma^2\big).$$

The function $F(k,\gamma)$ is a U-shaped quadratic function in $k$, and it has a minimum at

$$k_{min} = \frac{\gamma\big(30+50\gamma+23\gamma^2+2\gamma^3\big)}{2(48+114\gamma+88\gamma^2+23\gamma^3+\gamma^4)}.$$

It can be easily shown that

$$k_{min} - 2 = -\frac{192+426\gamma+302\gamma^2+69\gamma^3+2\gamma^4}{2(1+\gamma)(48+66\gamma+22\gamma^2+\gamma^3)} < 0.$$

and

$$F(2,\gamma) = 192+396\gamma+256\gamma^2+51\gamma^3+\gamma^4 > 0, \quad \forall\, \gamma \geq 0.$$

Therefore, $F(k,\gamma) > 0$ for all $k \geq 2$ and $\gamma \geq 0$. This means that $\left.\frac{\partial^2 \Psi(k,n,\gamma)}{\partial n^2}\right|_{n=k} < 0$, and thus for any $n > k$, $\frac{\partial^2 \Psi(k,n,\gamma)}{\partial n^2} < 0$. So the function $\Psi(k,n,\gamma)$ is concave in $n$ and the first derivative $\frac{\partial \Psi(k,n,\gamma)}{\partial n}$ is decreasing in $n$.

Furthermore, evaluating at $n = k$ yileds

$$\left.\frac{\partial \Psi(k,n,\gamma)}{\partial n}\right|_{n=k} = -k^3(64+148\gamma+108\gamma^2+25\gamma^3+\gamma^4) + k^2\gamma(60+92\gamma+37\gamma^2+3\gamma^3) - k\gamma^2(16+15\gamma+3\gamma^2) + \gamma^3(1+\gamma) \equiv f(k,\gamma),$$

with

$$\frac{\partial f(k,\gamma)}{\partial k} = -3k^2(64+148\gamma+108\gamma^2+25\gamma^3+\gamma^4)+2k\gamma(60+92\gamma+37\gamma^2+3\gamma^3) \\ -\gamma^2(16+15\gamma+3\gamma^2),$$

which is an inverted U-shaped quadratic function in $k$ and has a maximum at

$$k_{max} = \frac{\gamma(60+92\gamma+37\gamma^2+3\gamma^3)}{3(64+148\gamma+108\gamma^2+25\gamma^3+\gamma^4)} < 2.$$

Since

$$f(2,\gamma) = -(512+944\gamma+528\gamma^2+81\gamma^3+\gamma^4) < 0, \quad \forall\, \gamma \geq 0,$$

it follows that for any $k > 2$, $f(k,\gamma) < 0$, and thus $\frac{\partial \Psi(k,n,\gamma)}{\partial n}\Big|_{n=k} < 0$. Given that $\frac{\partial \Psi(k,n,\gamma)}{\partial n}$ is decreasing in $n$, we must have that for any $n > k$, $\frac{\partial \Psi(k,n,\gamma)}{\partial n} < 0$. This in turn means that $\Psi(k,n,\gamma)$ is a decreasing function in $n$. Combined with the condition $\Psi(k,k,\gamma) < 0$, it yields that $\Psi(k,n,\gamma) < 0$ for any $n > k$. So we have proved that for any $\gamma \geq 0$ and $2 \leq k \leq n$, $\Phi(\frac{1}{k-1}) < 0$. Combined with the fact that $\Phi(0) = \phi_0 < 0$ and $\Phi(v)$ is linear in $v$, it must hold that

$$\Delta W(k,n,v,\gamma) < 0, \quad \forall\, \gamma \geq 0, 2 \leq k \leq n, 0 < v < \frac{1}{k-1}.$$

□